

Differential Quadrature Method for Solving Hyperbolic Heat Conduction Problems

Ming-Hung Hsu

*Department of Electrical Engineering, National Penghu University,
Penghu, Taiwan 880, R.O.C.*

Abstract

This work analyzes hyperbolic heat conduction problems using the differential quadrature method. Numerical results are compared with published results to assess the efficiency and systematic procedure of this novel approach to solving hyperbolic heat conduction problems. The computed solutions to the hyperbolic heat conduction problems correlate well with published data. Two examples are analyzed using the proposed method. The effects of the relaxation time β on the temperature distribution are investigated. Hyperbolic heat conduction problems are solved efficiently using the differential quadrature method.

Key Words: Hyperbolic Heat Conduction, Differential Quadrature Method, Numerical Analysis

1. Introduction

Hyperbolic heat conduction has many engineering applications. Hyperbolic heat conduction equations have received considerable interest. Solutions to hyperbolic heat conduction problems can be found in a number of publications [1–29]. Yang [1] introduced the sequential method for determining boundary conditions in inverse hyperbolic heat conduction problems. Lor and Chu [2] presented the effect of interface thermal resistance on heat transfer in a composite medium using the thermal wave model. Antaki [3] investigated non-Fourier heat conduction in solid-phase reactions. Chen [4] proposed a new hybrid method and applied it to investigate the effect of the surface curvature of a solid body on hyperbolic heat condition. Sanderson et al. [5] investigated hyperbolic heat equations in laser generated ultrasound models. Liu et al. [6] discussed non-Fourier effects on the transient temperature response in a semitransparent medium caused by a laser pulse. Mullis [7,8] presented the rapid solidification and the propagation of heat at a finite velocity. Sahoo and Roetzel [9] proposed the hy-

perbolic axial dispersion model for heat exchangers. Roetzel et al. [10,11] analyzed the hyperbolic axial dispersion model and its application to a plate heat exchanger. Lin [12] studied the non-Fourier effect on fin performance under periodically varying thermal conditions. Abdel-Hamid [13] modeled non-Fourier heat conduction with a periodic thermal oscillation using the finite integral transform. Tang and Araki [14] investigated non-Fourier heat conduction in a finite medium under a periodic thermal disturbance of the surface. Tan and Yang [15,16] studied the propagation of thermal waves in transient heat conduction in a thin film. Weber [17] used a second-order explicit difference equation to analyze the ill-posed inverse heat conduction problem. Al-Khalidy [18,19] presented the solution to parabolic and hyperbolic inverse heat conduction problems. Chen et al. [20] used a second-order explicit difference equation to analyze hyperbolic inverse heat conduction problems. Beck et al. [21] proposed a sequential method, combined with a concept of future time, to solve the ill-posed inverse heat conduction problem. Yang [22,23] determined temperature-dependent thermo physical properties from measured temperature responses and boundary conditions of the medium. Carey and Tsai [24] investigated

*Corresponding author. E-mail: hsu@npu.edu.tw

hyperbolic heat transfer with reflection. Glass et al. [25] and Yeung and Tung [26] studied the numerical solution of the hyperbolic heat conduction problem and elucidated the effect of the surface radiation on thermal wave propagation in a one-dimensional slab. Anderson et al. [27] applied computational fluid mechanics and elucidated heat transfer using various numerical methods. Tamma and Raildar [28] proposed an tailored trans-finite-element formulation for hyperbolic heat conduction, incorporating non-Fourier effects. Yang [29] analyzed the two-dimensional hyperbolic heat conduction using a high-resolution numerical method. Chen and Lin [30,31] solved two-dimensional hyperbolic heat conduction problems using the hybrid numerical scheme.

This work attempts to solve hyperbolic heat conduction problems using the proposed differential quadrature method. To our knowledge, this work is the first to solve hyperbolic heat conduction problems using this method. The Chebyshev-Gauss-Lobatto point distribution is utilized. The integrity and computational accuracy of the differential quadrature method in solving this problem are demonstrated through various case studies.

2. Differential Quadrature Method

Differential quadrature was originally developed by simple analogy with integral quadrature, which is derived using the interpolation function [32–34]. Currently, many engineers apply numerical techniques to solve sets of linear algebraic equations. They include the Rayleigh-Ritz method, the Galerkin method, the finite element method, the boundary element method and others, which are applied to solve differential equations. Bellman et al. [32,33] first introduced the differential quadrature method. His work has been applied in diverse areas of computational mechanics and many researchers have claimed that the differential quadrature approach is a highly accurate scheme that requires minimal computational efforts. Differential quadrature has been shown to be a powerful tool for solving initial and boundary value problems, and has thus become an alternative to existing methods. Bert et al. [34–46], who investigated the static and free vibration of beams and rectangular plates using the differential quadrature method, proposed the δ technique that can be applied to the double boundary conditions of plate and beam problems. In this approach,

boundary points are chosen at a small distance. However, δ must be small to ensure the accuracy of the solution, but solutions oscillate when δ is excessively small. Bert et al. [34–46] incorporated boundary conditions using weighted coefficients. In the formulation of the differential quadrature approach, multiple boundary conditions are directly applied to the weighted coefficients and unlike in the δ -interval method, no point need be selected. The accuracy of calculated results will be independent of the δ -interval. Liew et al. [47–50], who analyzed rectangular plates at rest on Winkler foundations using the differential quadrature method, also presented a static analysis of laminated composite plates subjected to transverse loads using the differential quadrature method. The efficiency and accuracy of the Rayleigh-Ritz method depend on the number and accuracy of selected comparison functions. However, the differential quadrature method does not generate such difficulties when appropriate comparison functions are used. The differential quadrature method is a continuous function that can be approximated by a high-order polynomial in the overall domain, and a derivative of a function at a sample point can be approximated as a weighted linear summation of functional values at all sample points in the overall domain of that variable. Using this approximation, the differential equation is then transformed into a set of algebraic equations. The number of equations depends on the number of sample points selected. As for any polynomial approach, increasing the number of sample points increases the accuracy of the solution. Numerical interpolation schemes can be used to eliminate potential oscillations in numerical results that are associated with high-order polynomials. For a function, $f(x, t)$, the differential quadrature method approximation for the m^{th} order derivative at the i^{th} sample point is given as follows [47–50].

$$\frac{\partial^m}{\partial x^m} \begin{Bmatrix} f(x_1, t) \\ f(x_2, t) \\ \vdots \\ f(x_N, t) \end{Bmatrix} \cong [D_{ij}^{(m)}] \begin{Bmatrix} f(x_1, t) \\ f(x_2, t) \\ \vdots \\ f(x_N, t) \end{Bmatrix} \quad \text{for } i, j = 1, 2, \dots, N \quad (1)$$

where $f(x_i, t)$ is the functional value at grid point x_i , and

$D_{ij}^{(m)}$ is the weighting coefficient of the m^{th} order differentiation of these functional values. The most convenient technique is to choose equally spaced grid points, but the numerical results were inaccurate in this work when equally spaced points were selected. This finding demonstrates that the choice of a grid point distribution and the test functions markedly influence the efficiency and accuracy of the results in some cases. The selection of grid points always importantly affects solution accuracy. Unequally spaced sample points on each domain have the following Chebyshev-Gauss-Lobatto distribution in the present computation [47–50].

$$x_i = \frac{1}{2} \left(1 - \cos \frac{(i-1)\pi}{N-1} \right) \quad \text{for } i = 1, 2, \dots, N \quad (2)$$

The differential quadrature weighted coefficients can be derived using numerous techniques. To overcome the numerically poor conditions in determining the weighted coefficients, $D_{ij}^{(m)}$, the following Lagrangian interpolation polynomial is introduced [47–50].

$$f(x, t) \cong \sum_{i=1}^N \frac{M(x)}{(x - x_i)M_1(x_i)} f(x_i, t) \quad (3)$$

where

$$M(x) = \prod_{j=1}^N (x - x_j)$$

$$M_1(x_i) = \prod_{j=1, j \neq i}^N (x_i - x_j) \quad \text{for } i = 1, 2, \dots, N$$

Substituting Eq. (3) into Eq. (1) yields the following equations [47–50].

$$D_{ij}^{(1)} = \frac{M_1(x_i)}{(x_i - x_j)M_1(x_j)} \quad \text{for } i, j = 1, 2, \dots, N \quad \text{and } i \neq j \quad (4)$$

and

$$D_{ii}^{(1)} = - \sum_{j=1, j \neq i}^N D_{ij}^{(1)} \quad \text{for } i = 1, 2, \dots, N \quad (5)$$

Once the grid points are selected, the coefficients of

the weighted matrix can be acquired using Eqs. (4) and (5). Notably, the numbers of test functions exceed the highest order of the derivative in the governing equations. High-order derivatives of weighted coefficients can also be acquired using matrix multiplication [47–50], as follows.

$$D_{ij}^{(2)} = \sum_{k=1}^N D_{ik}^{(1)} D_{kj}^{(1)} \quad \text{for } i, j = 1, 2, \dots, N \quad (6)$$

$$D_{ij}^{(3)} = \sum_{k=1}^N D_{ik}^{(1)} D_{kj}^{(2)} \quad \text{for } i, j = 1, 2, \dots, N \quad (7)$$

$$D_{ij}^{(4)} = \sum_{k=1}^N D_{ik}^{(1)} D_{kj}^{(3)} \quad \text{for } i, j = 1, 2, \dots, N \quad (8)$$

3. Analysis

Consider a finite slab and a dimensionless formulation; the hyperbolic equation for a one-dimensional problem has the form [1]

$$\frac{\partial^2 T(x, t)}{\partial t^2} + 2 \frac{\partial T(x, t)}{\partial t} - \frac{\partial^2 T(x, t)}{\partial x^2} = 0 \quad \text{at } t > 0, \quad 0 < x < l \quad (9)$$

with the initial condition,

$$T(x, 0) = 0 \quad (10)$$

$$\frac{\partial T(x, t)}{\partial t} = 0 \quad (11)$$

and the boundary conditions,

$$T(0, t) = 1 \quad (12)$$

$$\frac{\partial T(l, t)}{\partial x} = 0 \quad (13)$$

where T represents the temperature field $T(x, t)$. The differential quadrature method is applied and Eq. (1) is substituted into Eq. (9). The equation discretizes the sample points as,

$$\begin{aligned} & [1] \left\{ \frac{\partial^2 T(x_i, t)}{\partial t^2} \right\} + [2] \left\{ \frac{\partial T(x_i, t)}{\partial t} \right\} \\ & - [D_{i1}^{(2)} \quad D_{i2}^{(2)} \quad \dots \quad D_{iN}^{(2)}] \{T(x_j, t)\} = \{0\} \end{aligned} \quad (14)$$

for $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, N$

Equation (12) can be rewritten as

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \{T(x_j, t)\} = \{1\} \quad (15)$$

for $j = 1, 2, \dots, N$

According to the differential quadrature method, Eq. (13) takes the following discrete forms;

$$\begin{bmatrix} \frac{D_{N1}^{(1)}}{l} & \frac{D_{N2}^{(1)}}{l} & \cdots & \frac{D_{NN}^{(1)}}{l} \end{bmatrix} \{T(x_j, t)\} = \{0\} \quad (16)$$

for $j = 1, 2, \dots, N$

Many inner and boundary points in the computational domain contribute directly to the calculation of the derivatives and the state variables, due to the global nature of the differential quadrature method. Many inner and boundary points in the computational domain contribute directly to the calculation of the derivatives and the state variables due to the global nature of the differential quadrature method. Figures 1–3 plot the temperature distributions at $t = 0.5, 1.0$ and 1.5 , respectively. The numerical results calculated using the differential quadrature method are very consistent with the results in the literature [1] for $t = 0.5, 1.0$ and 1.5 . Consider a slab with

thickness l and constant thermal properties. This slab originally has a uniform temperature distribution [1]. The adiabatic condition is applied to $x = 0$. At a specific time $t = 0$, a heat flux $q(t)$ is applied to $x = l$. A mathematical formation of the problem is as follows [1].

$$\beta \rho C \frac{\partial^2 T(x, t)}{\partial t^2} + \rho C \frac{\partial T(x, t)}{\partial t} - k \frac{\partial^2 T(x, t)}{\partial x^2} = 0 \quad (17)$$

at $t > 0, 0 < x < l$

The initial conductions are,

$$T(x, 0) = T_0 \quad 0 \leq x \leq l \quad (18)$$

$$\frac{\partial T(x, t)}{\partial t} = 0 \quad \text{at } t = 0, 0 \leq x \leq l \quad (19)$$

and the boundary conditions are,

$$\frac{\partial T(x, t)}{\partial x} = 0 \quad \text{at } x = 0, t > 0 \quad (20)$$

$$\beta \frac{\partial q(x, t)}{\partial t} + k \frac{\partial T(x, t)}{\partial x} + q(x, t) = 0 \quad \text{at } x = l, t > 0 \quad (21)$$

where T represents the temperature field $T(x, t)$, k is the thermal conductivity, ρ is the density, C is the specific heat, ρC is the heat capacity per unit volume, and β is the relaxation time, which is nonnegative. The differential quadrature method is applied and Eq. (1) is substituted into Eq. (17). The equation discretizes the sample points as

$$\begin{aligned} & [\beta \rho C] \left\{ \frac{\partial^2 T(x_i, t)}{\partial t^2} \right\} + [\rho C] \left\{ \frac{\partial T(x_i, t)}{\partial t} \right\} \\ & - \left[k \frac{D_{i1}^{(2)}}{l^2} \quad k \frac{D_{i2}^{(2)}}{l^2} \quad \cdots \quad k \frac{D_{iN}^{(2)}}{l^2} \right] \{T(x_j, t)\} = \{0\} \end{aligned} \quad (22)$$

for $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, N$

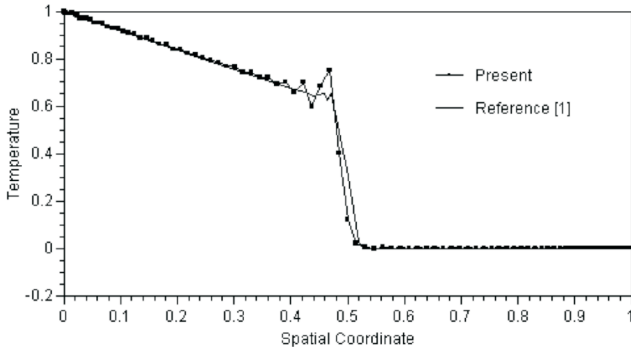


Figure 1. Temperature distribution at $t = 0.5$.

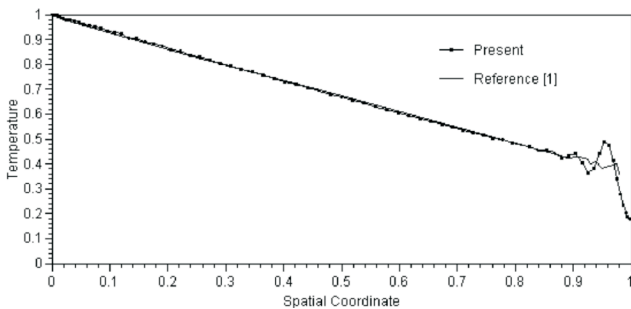


Figure 2. Temperature distribution at $t = 1.0$.

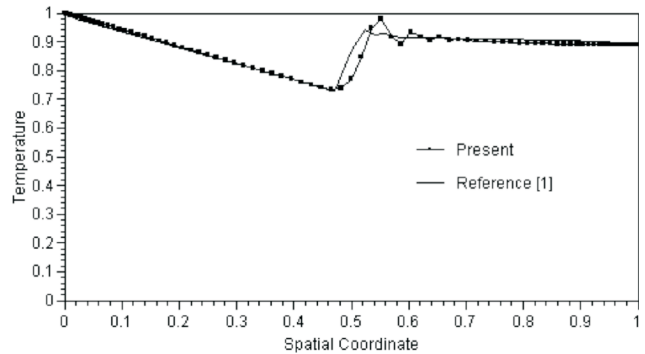


Figure 3. Temperature distribution at $t = 1.5$.

Equation (20) can be rewritten as

$$\left[\frac{D_{1,1}^{(1)}}{l} \quad \frac{D_{1,2}^{(1)}}{l} \quad \frac{D_{1,3}^{(1)}}{l} \quad \dots \quad \frac{D_{1,N-1}^{(1)}}{l} \quad \frac{D_{1,N}^{(1)}}{l} \right] \{T(x_j, t)\} = \{0\}$$

for $j = 1, 2, \dots, N$

(23)

In the differential quadrature method, Eq. (21) takes the following discrete forms:

$$[\beta] \left\{ \frac{\partial T(x_N, t)}{\partial t} \right\} + \left[k \frac{D_{N1}^{(1)}}{l} \quad k \frac{D_{N2}^{(1)}}{l} \quad \dots \quad k \frac{D_{NN}^{(1)}}{l} \right] \{T(x_j, t)\} + \{q(x_N, t)\} = \{0\} \quad \text{for } j = 1, 2, \dots, N$$

(24)

Many inner and boundary points in the computational domain are used directly in the calculation of the derivatives and the state variables due to the global nature of the differential quadrature method. Consider a slab with thickness $l = 0.035 \text{ m}$ and constant thermal properties $k = 50 \text{ W/m}^2$ and $k/\rho C = 0.0001327 \text{ m}^2/\text{sec}$ [1]. Figure 4 plots the input heat flux [1]. A time-varying heat flux is applied at the side $x = l$ between time steps 200 and 230 and the magnitude of the heat flux varies at each time step. Figures 5–9 depict the direct solution when $\beta = 0, 1, 10, 100$, and 1000 . The values of l, k, ρ and C are set to unity in Eq. (17). The strength of the heat source is presented as a time-varying function. Figures 5–9 plot the numerical results between time steps 200 to 230. The curvature increases and reduces the temperature, depending on the relaxation time β . The speed of heat propagation varies with the relaxation time β . The magnitude of the thermal properties significantly affects the analysis of the temperature distribution. The numerical results show that the temperature wave is sensitive to the

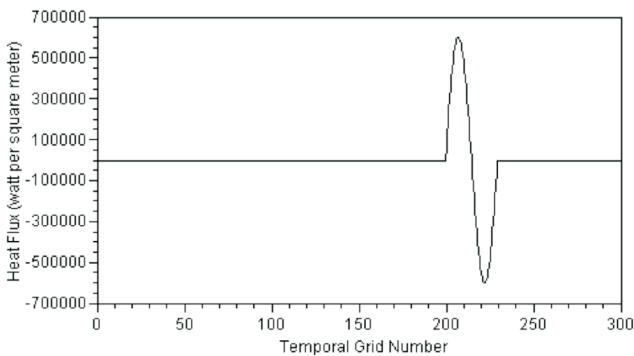


Figure 4. Time-varying heat flux is applied at the side $x = l$ [1].

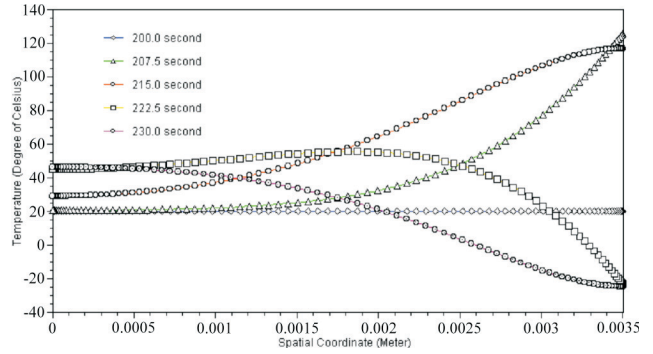


Figure 5. Temperature distribution when relaxation time $\beta = 0$.

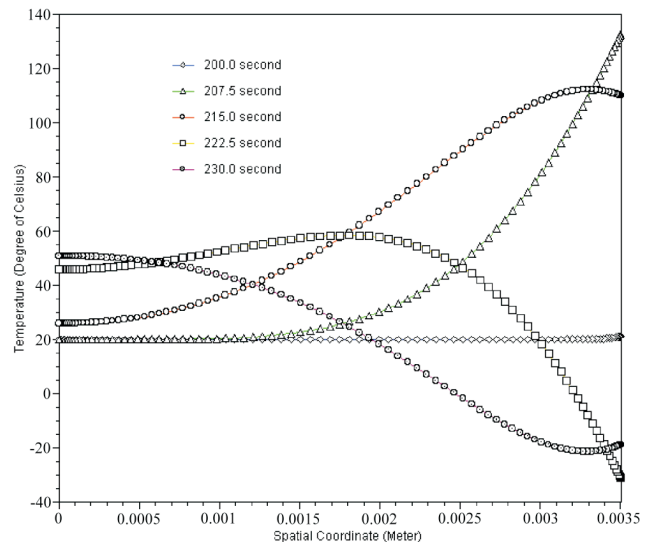


Figure 6. Temperature distribution when relaxation time $\beta = 1$.

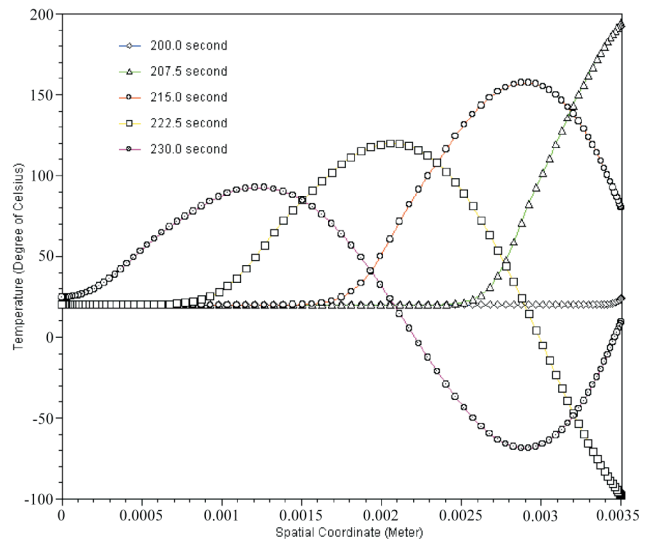


Figure 7. Temperature distribution when relaxation time $\beta = 10$.

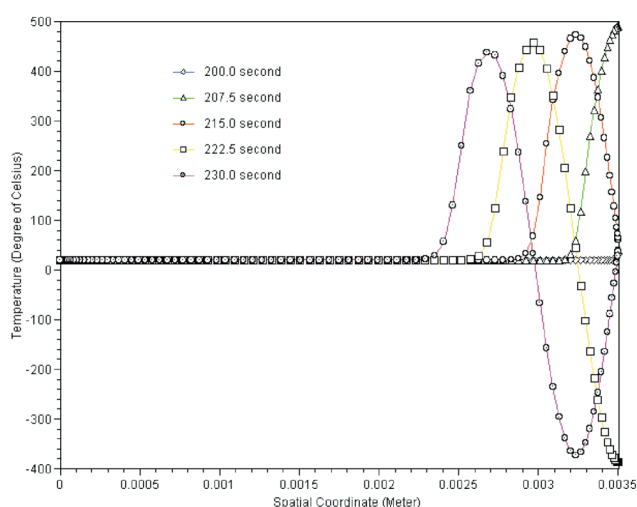


Figure 8. Temperature distribution when relaxation time $\beta = 100$.

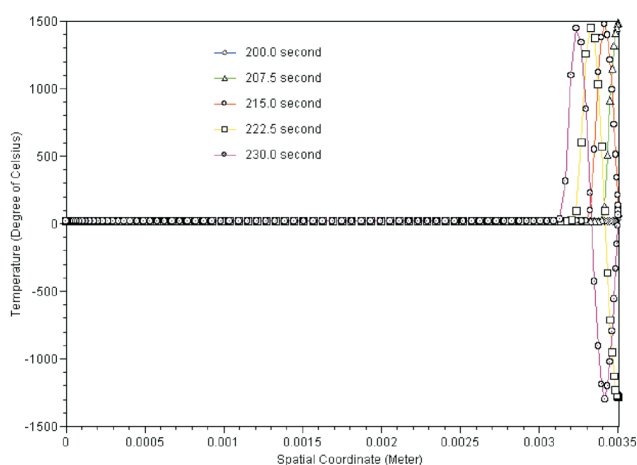


Figure 9. Temperature distribution when relaxation time $\beta = 1000$.

relaxation time. The first investigation yields good results concerning the use of the differential quadrature method for solving hyperbolic heat conduction problems.

4. Conclusion

This work proposes the differential quadrature method for solving hyperbolic heat conduction problems. The partial differential equation is reduced to a set of algebraic equations using the differential quadrature method. The considered examples demonstrate that the present method is in good agreement with solutions taken from the literature [1]. Simulation results verify that the differential quadrature method yields accurate results with relatively little computational and modeling effort.

The magnitude of the thermal properties significantly influences the temperature distribution. The relaxation time β affects the temperature distribution. The demonstrated simplicity and accuracy of this formulation makes it a good candidate for modeling various problems in science and engineering.

References

- [1] Yang, C. Y., "Direct and Inverse Solutions of Hyperbolic Heat Conduction Problems," *Journal of Thermophysics and Heat Transfer*, Vol. 19, pp. 217–225 (2005).
- [2] Lor, W. B. and Chu, H. S., "Effect of Interface Thermal Resistance on Heat Transfer in a Composite Medium Using the Thermal Wave Model," *Internal Journal of Heat and Mass Transfer*, Vol. 43, pp. 653–663 (2000).
- [3] Antaki, P. J., "Importance of Non-Fourier Heat Conduction in Solid-Phase Reactions," *Combustion and Flame*, Vol. 112, pp. 329–341 (1998).
- [4] Chen, T. M., "Numerical Solution of Hyperbolic Heat Conduction in Thin Surface Layers," *International Journal of Heat and Mass Transfer*, Vol. 50, pp. 4424–4429 (2007).
- [5] Sanderson, T., Ume, C. and Jarzynski, J., "Hyperbolic Heat Equations in Laser Generated Ultrasound Models," *Ultrasonics*, Vol. 33, pp. 415–421 (1995).
- [6] Liu, L. H., Tan, H. P. and Tong, T. W., "Non-Fourier Effects on Transient Temperature Response in Semi-transparent Medium Caused by Laser Pulse," *International Journal of Heat and Mass Transfer*, Vol. 44, pp. 3335–3344 (2001).
- [7] Mullis, A. M., "Rapid Solidification and a Finite Velocity for the Propagation of Heat," *Material Science and Engineering*, Vol. 226–28, pp. 28–32 (1997).
- [8] Mullis, A. M., "Rapid Solidification within the Framework of a Hyperbolic Conduction Model," *International Journal of Heat and Mass Transfer*, Vol. 40, pp. 4085–4094 (1997).
- [9] Sahoo, R. K. and Roetzel, W., "Hyperbolic Axial Dispersion Model for Heat Exchangers," *International Journal of Heat and Mass Transfer*, Vol. 45, pp. 1261–1270 (2002).
- [10] Roetzel, W. and Das, S. K., "Hyperbolic Axial Dispersion Model: Concept and Its Application to a Plate Heat Exchanger," *International Journal of Heat and*

- Mass Transfer*, Vol. 38, pp. 3065–3076 (1995).
- [11] Roetzel, W. and Na Ranong, C., “Consideration of Maldistribution in Heat Exchangers Using the Hyperbolic Dispersion Model,” *Chemical Engineering and Processing*, Vol. 38, pp. 675–681 (1999).
 - [12] Lin, J. Y., “The Non-Fourier Effect on the Fin Performance under Periodic Thermal Conditions,” *Applied Mathematical Modelling*, Vol. 22, pp. 629–640 (1998).
 - [13] Abdel-Hamid, B., “Modelling Non-Fourier Heat Conduction with Periodic Thermal Oscillation Using the Finite Integral Transform,” *Applied Mathematical Modelling*, Vol. 23, pp. 899–914 (1999).
 - [14] Tang, D. W. and Araki, N., “Non-Fourier Heat Conduction in a Finite Medium under Periodic Surface Thermal Disturbance,” *International Journal of Heat and Mass Transfer*, Vol. 39, pp. 1585–1590 (1996).
 - [15] Tan, Z. M. and Yang, W. J., “Propagation of Thermal Waves in Transient Heat Conduction in a Thin Film,” *Journal of the Franklin Institute*, Vol. 336, pp. 185–197 (1999).
 - [16] Tan, Z. M. and Yang, W. J., “Heat Transfer during Asymmetrical Collision of Thermal Waves in a Thin Film,” *International Journal of Heat and Mass Transfer*, Vol. 40, pp. 3999–4006 (1997).
 - [17] Weber, C., “Analysis and Solution of the Ill-Posed Inverse Heat Conduction Problem,” *International Journal of Heat and Mass Transfer*, Vol. 24, pp. 1783–1791 (1981).
 - [18] Al-Khalidy, N., “On the Solution of Parabolic and Hyperbolic Inverse Heat Conduction Problems,” *International Journal of Heat and Mass Transfer*, Vol. 41, pp. 3731–3740 (1998).
 - [19] Al-Khalidy, N., “Analysis of Boundary Inverse Heat Conduction Problems Using Space Marching with Savitzky-Golay Digital Filter,” *International Communication of Heat and Mass Transfer*, Vol. 26, pp. 199–208 (1999).
 - [20] Chen, H. T., Peng, S. Y. and Fang, L. C., “Numerical Method for Hyperbolic Inverse Heat Conduction Problems,” *International Communication of Heat and Mass Transfer*, Vol. 28, pp. 847–856 (2001).
 - [21] Beck, J. V., Blackwell, B. and Clair, C. R. S., *Inverse Heat Conduction—Ill Posed Problem*, First ed., Wiley, New York (1985).
 - [22] Yang, C. Y., “Determination of Temperature-Dependent Thermophysical Properties from Temperature Responses Measured and Medium’s Boundaries,” *International Journal of Heat and Mass Transfer*, Vol. 43, pp. 1261–1270 (2000).
 - [23] Yang, C. Y., “Estimation of Boundary Conditions in Nonlinear Inverse Heat Conduction Problems,” *Journal of Thermophysics and Heat Transfer*, Vol. 17, pp. 389–395 (2003).
 - [24] Carey, G. F. and Tsai, M., “Hyperbolic Heat Transfer with Reflection” *Numerical Heat Transfer*, Vol. 5, pp. 309–327 (1982).
 - [25] Glass, D. E., Ozisik, M., McRae, D. S. and Vick, B., “On the Numerical Solution of Hyperbolic Heat Conduction,” *Numerical Heat Transfer*, Vol. 8, pp. 497–504 (1985).
 - [26] Yeung, W. K. and Tung, T. L., “A Numerical Scheme for Non-Fourier Heat Condition, Part I: One-Dimensional Problem Formulation and Applications,” *Numerical Heat Transfer*, Vol. 33, pp. 215–233 (1998).
 - [27] Anderson, D. A., Tannehill, J. C. and Pletcher, R. H., *Computational Fluid Mechanics and Heat Transfer*, Hemisphere, Washington, DC (1984).
 - [28] Tamma, K. K. and Raildar, S. B., “Specially Tailored Trans-Finite-Element Formulations for Hyperbolic Heat Conduction Involving Non-Fourier Effects,” *Numerical Heat Transfer, Part B*, Vol. 15, pp. 211–226 (1989).
 - [29] Yang, H. Q., “Solution of Two-Dimensional Hyperbolic Heat Conduction by High-Resolution Numerical Method,” *Numerical Heat Transfer*, Vol. A21, pp. 333–349 (1992).
 - [30] Chen, H. T. and Lin, J. Y., “Numerical Analysis for Hyperbolic Heat Conduction,” *International Journal of Heat and Mass Transfer*, Vol. 36, pp. 2891–2898 (1993).
 - [31] Chen, H. T. and Lin, J. Y., “Analysis of Two-Dimensional Hyperbolic Heat Conduction Problems,” *International Journal of Heat and Mass Transfer*, Vol. 37, pp. 153–164 (1994).
 - [32] Bellman, R. and Casti, J., “Differential Quadrature and Long-Term Integration,” *Journal of Mathematical Analysis and Application*, Vol. 34, pp. 235–238 (1971).
 - [33] Bellman, R. E., Kashef, B. G. and Casti, J., “Differential Quadrature: A Technique for Rapid Solution of Nonlinear Partial Differential Equations,” *Journal Computational Physics*, Vol. 10, pp. 40–52 (1972).
 - [34] Bert, C. W. and Mailk, M., “Differential Quadrature

- Method in Computational Mechanics: A Review," *Applied Mechanics Review*, Vol. 49, pp. 1–28 (1996).
- [35] Bert, C. W., Jang, S. K. and Striz, A. G., "Two New Approximate Methods for Analyzing Free Vibration of Structural Components," *AIAA Journal*, Vol. 26, pp. 612–618 (1988).
- [36] Bert, C. W. and Malik, M., "Free Vibration Analysis of Tapered Rectangular Plates by Differential Quadrature Method: A Semi-Analytical Approach," *Journal of Sound and Vibration*, Vol. 190, pp. 41–63 (1996).
- [37] Bert, C. W. and Malik, M., "On the Relative Effects of Transverse Shear Deformation and Rotary Inertia on the Free Vibration of Symmetric Cross-Ply Laminated Plates," *Journal of Sound and Vibration*, Vol. 193, pp. 927–933 (1996).
- [38] Bert, C. W., Wang, X. and Striz, A. G., "Differential Quadrature for Static and Free Vibration Analysis of Anisotropic Plates," *International Journal of Solids and Structures*, Vol. 30, pp. 1737–1744 (1993).
- [39] Bert, C. W., Wang, X. and Striz, A. G., "Convergence of the DQ Method in the Analysis of Anisotropic Plates," *Journal of Sound and Vibration*, Vol. 170, pp. 140–144 (1994).
- [40] Striz, A. G., Chen, W. and Bert, C. W., "Static Analysis of Structures by the Quadrature Element Method (QEM)," *International Journal of Solids and Structures*, Vol. 31, pp. 2807–2818 (1994).
- [41] Jang, S. K., Bert, C. W. and Striz, A. G., "Application of Differential Quadrature to Static Analysis of Structural Components," *International Journal for Numerical Methods in Engineering*, Vol. 28, pp. 561–577 (1989).
- [42] Kang, K., Bert, C. W. and Striz, A. G., "Static Analysis of a Curved Shaft Subjected to End Torques," *International Journal of Solids and Structures*, Vol. 33, pp. 1587–1596 (1996).
- [43] Wang, X. and Bert, C. W., "A New Approach in Applying Differential Quadrature to Static and Free Vibrational Analyses of Beams and Plates," *Journal of Sound and Vibration*, Vol. 162, pp. 566–572 (1993).
- [44] Malik, M. and Bert, C. W., "Implementing Multiple Boundary Conditions in the DQ Solution of Higher-Order PDE's Application to Free Vibration of Plates," *International Journal for Numerical Methods in Engineering*, Vol. 39, pp. 1237–1258 (1996).
- [45] Feng, Y. and Bert, C. W., "Application of the Quadrature Method to Flexural Vibration Analysis of a Geometrically Nonlinear Beam," *Nonlinear Dynamics*, Vol. 3, pp. 13–18 (1992).
- [46] Chen, W. L., Striz, A. G. and Bert, C. W., "High-Accuracy Plane Stress and Plate Elements in the Quadrature Element Method," *International Journal of Solid and Structures*, Vol. 37, pp. 627–647 (2000).
- [47] Liew, K. M., Han, J. B., Xiao, Z. M. and Du, H., "Differential Quadrature Method for Mindlin Plates on Winkler Foundations," *International Journal of Mechanical Sciences*, Vol. 38, pp. 405–421 (1996).
- [48] Han, J. B. and Liew, K. M., "Axisymmetric Free Vibration of Thick Annular Plates," *International Journal of Mechanical Science*, Vol. 41, pp. 1089–1109 (1999).
- [49] Liew, K. M., Han, J. B. and Xiao, Z. M., "Differential Quadrature Method for Thick Symmetric Cross-Ply Laminates with First-Order Shear Flexibility," *International Journal of Solids and Structures*, Vol. 33, pp. 2647–2658 (1996).
- [50] Du, H., Liew, K. M. and Lim, M. K., "Generalized Differential Quadrature Method for Buckling Analysis," *Journal of Engineering Mechanics*, Vol. 122, pp. 95–100 (1996).

Manuscript Received: Jan. 27, 2008

Accepted: Aug. 12, 2008