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An Approximation Solution for the Twin Prime Conjecture

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Abstract

In this study, we investigate the existence of numerous twin prime pairs according to the prime number inferred by the sieve of Eratosthenes. Given a number M = $(6n + 5)^2$, at least three twin prime pairs can be found from the incremental range, which is increased from $(6n + 5)^2$ to $[6(n + 1) + 5]^2$ for n = 0 to infinite. Thus, we might be able to prove the twin prime conjecture proposed by de Polignac in 1849, that is, several prime numbers p exist for each natural number k by denoting p + 2k as the prime number when k = 1. Instead of twin prime pairs occurring irregularly, we infer that the twin prime conjecture solution might solved by satisfying two conditions: (1) eliminating the nontwin prime pairs in associated twin prime pairs would be regular, and (2) the incremental range from $(6n + 5)^2$ to $[6(n + 1) + 5]^2$ for n = 0 to ∞ would be regular. These conditions may not have been considered in previous studies that explored the question on whether numerous twin prime pairs exist, which has been one of the open questions in number theory for more than a century.

***Key Words*:** Twin Primes, Number Theory, Prime Number, Incremental Range

1. Introduction

In this study, we investigate the existence of numerous twin prime pairs by employing the concept of inferring the prime number proposed by the sieve of Eratosthenes. While we set $(6n + 5)^2$ as the range for estimating twin prime pairs, we discover that at least three additional twin prime pairs as n (positive integer) are increased by 1. Thus, we confirm the twin prime conjecture proposed by de Polignac in 1849 by proving that numerous prime numbers p exist for each natural number k by denoting p + 2k as a prime number when k = 1.

In 1990, Hilbert raised 23 mathematical problems at the International Mathematical Conference held in Paris [1]. The twin prime puzzle (i.e., whether numerous twin primes exist) is still listed as the eighth problem, even though twin prime conjecture was proposed by de Polignac in 1849.

To the best of our understanding, only 3 questions remain completely unsolved among the 23 mathematical problems [1]. As for the recent studies on Hilbert's unsolved eighth problem (twin prime conjecture), Y. Zhang proved that bounded gaps among primes are all less than 70 million [2].

After the establishment of a bound of 70 million on the narrowest gap among primes [2], immense interest has emerged in employing Zhang's argument to lower the bound. T. Tao and several other mathematicians worked together to cut the bound down from 70 million to 246 via an online "polymath" project, as well as traditional research channels [3]. However, these efforts are based on the conjecture about prime intervals [4]. In other words, the proposition seems to have changed from whether nu-

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merous twin prime pairs exist to how big the gap is between two prime numbers [3,5,6].

Instead of narrowing down the gap between two prime numbers, this study focuses on whether numerous twin prime pairs exist, which has been one of the open questions in number theory for more than a century. While setting $(6n + 5)^2$ as the range for estimating twin prime pairs according to the sieve of Eratosthenes [7], we prove that at least three additional twin prime pairs would exist as n (positive integer) is increased by 1, that is, twin prime pairs would become numerous as n is increased from 1 to ∞.

2. Definition and Employment of Associated Twin Prime (ATP) Pairs

In this study, we define the pair (6n – 1, 6n + 1) as associated twin prime (ATP) pairs. As shown in Table 1, the first and second numbers in the ATP pairs would not be included in multiples of 2 and 3.

For convenience, we denote $ATPn^a$ and $ATPn^b$ as the first and second numbers of an ATP pair (6n – 1, 6n + 1), i.e., 6n – 1 and 6n + 1. As for any natural number starting from 4, any multiple of 2 or 3 would be excluded as an ATP pair; examples include the natural numbers shown in Columns 1, 3, 5, and 6 of Table 1. ATP pairs are presented in Columns 2 and 4 of Table 1 because $ATPn^a$ and $ATPn^b$ in the ATP pairs (ATPn) would be (6n – 1, 6n + 1).

Thus, if a natural number is not divisible by 2 or 3, then the natural number is presented as 6n ± 1. As a result, prime numbers, except for 2 and 3, are presented as 6n ± 1, where n is a positive integer. The difference between 6n – 1 and 6n + 1 is exactly 2, i.e., the pair (6n + 1, 6n – 1) for n = 1 to ∞ is regarded as an ATP pair.

The ATP pair encompasses twin prime pairs, such as (5, 7) when n = 1, (11, 13) when n = 2, (17, 19) when n = 3, (29, 31) when n = 5, (41, 43) when n = 7, and (59, 61) when n = 10, except for (3, 5). However, the ATP pair (6n – 1, 6n + 1) is not a twin prime pair for any n (positive integer), as in the following: (23, 25) when n = 4, (35, 37) when n = 6, (47, 49) when n = 8, and (53, 55) when n = 9. Therefore, for the ATP pairs (6n – 1, 6n + 1), when n is from 1 to 10, we conclude that the twin prime pairs are ATP pairs except for (3, 5); however, ATP pairs are not

Table 1. ATP pairs (6n – 1, 6n + 1)

n	Column 1	Column 2	Column 3	Column 4	Column 5	Column 6
		$ATPn^a$		$ATPn^b$		
	6n – 2	6n – 1	6n	6n + 1	6n + 2	6n + 3
n = 1	4	5	6	7	8	9
n = 2	10	11	12	13	14	15
n = 3	16	17	18	19	20	21
n = 4	22	23	24	25	26	27
n = 5	28	29	30	31	32	33
n = 6	34	35	36	37	38	39
n = 7	40	41	42	43	44	45
n = 8	46	47	48	49	50	51
n = 9	52	53	54	55	56	57
n = 10	58	59	60	61	62	63
n = 11	64	65	66	67	68	69
n = 12	70	71	72	73	74	75
n = 13	76	77	78	79	80	81
n = 14	82	83	84	85	86	87
n = 15	88	89	90	91	92	93
n = 16	94	95	96	97	98	99
n = 17	100	101	102	103	104	105
⋮	⋮	⋮	⋮	⋮	⋮	⋮

all twin prime pairs. In other words, if and only if $ATPn^a$ and $ATPn^b$ are prime numbers, then ATPn is regarded as a twin prime pair.

We therefore find evidence of the existence of numerous twin prime pairs because they continuously increase as the selected incremental range expands. The selection of the incremental range is explained in the following section.

3. Selection of Incremental Range

By calculating the number of ATP pairs within a certain range, we can explain why the number of ATP pairs increases as the range is increased incrementally, i.e., from $(6n + 5)^2$ to $[6(n + 1) + 5]^2$ when n = 1 to ∞. We argue that the number of twin prime pairs would increase infinitely when the selected rangeis expanded infinitely. Thus, the twin prime conjecture is confirmed in this study.

According to the sieve of Eratosthenes, N is confirmed as a prime number if N is not divisible by a prime number not greater than $\sqrt{N}$. Thus, when $\sqrt{N} = 6n - 1$ (prime number) or $\sqrt{N} = 6n + 1$ (prime number), N is a prime number close to $(6n + 5)^2$, i.e., $[6(n + 1) - 1]^2 = N$, which could not be divided by any prime number smaller than $\sqrt{N}$, such as 6n + 1. This result is due to 6n + 1 being smaller than 6n + 5, i.e., $6n + 1 < 6n + 5 = \sqrt{N}$.

As mentioned above, if and only if $ATPn^a$ and $ATPn^b$ are all prime numbers, then ATPn is regarded as a twin prime pair. Thus, twin prime pairs can be estimated if we could examine whether all $ATPn^a$ and $ATPn^b$ less than $(6n + 5)^2$, i.e., $[ATP(n + 1)^a]^2 = [6(n + 1) - 1]^2$, could not be divided by any prime number less than $ATP(n + 1)^a$, such as 6n + 1 less than 6n + 5, i.e., 6(n + 1) – 1. Thus, the following context denotes $(6n + 5)^2$ as the selected incremental range in this study. This range is convenient for verifying the existence of unlimited twin prime pairs.

While estimating twin prime pairs for the range smaller than $(6n + 5)^2$ according to the sieve of Eratosthenes, we calculate the number of ATP pairs for the range of $(6n + 5)^2$ for n when n = 1 to ∞ (ATPGn). ATPGn includes $\left\{\frac{[6n+5]^2-1}{6}-1\right\}$ ATP pairs, which are calculated as a cycle of every six integers; that is, ATPG1 (i.e., ATPGn when n = 1) would have 19 ATP pairs when the range is less than $(6n + 5)^2 = 121$ (i.e., n = 1). Similarly, ATPG2 = 47 when the range is less than $(6n + 5)^2 = 289$ (i.e., n = 2), ATPG3 = 87 when the range is less than $(6n + 5)^2 = 529$ (i.e., n = 3), ATPG4 = 139 when the range is less than $(6n + 5)^2 = 841$ (i.e., n = 4), and so on.

Thus, if the twin prime numbers for ATPGn are increased by more than 1 if incremented by 1 each time, then we can prove the twin prime conjecture.

We then calculate the increasing rate of (ATPGn) (ATPGn%) as n is incremented by 1. The formula is shown as follows.

$$
\begin{aligned}
ATPGn\% &= \left\{\frac{[ATP(n+1)^a]^2-1}{6}-1\right\} \Big/ \left\{\frac{[ATP(n)^a]^2-1}{6}-1\right\} \\
&= \left\{\frac{[6(n+1)-1]^2-1}{6}-1\right\} \Big/ \left\{\frac{[6(n)-1]^2-1}{6}-1\right\} \\
&= \left\{\frac{[6n+5]^2-1}{6}-1\right\} \Big/ \left\{\frac{[6n-1]^2-1}{6}-1\right\} \\
&= \left\{\frac{36n^2+60n+25-1}{6}-1\right\} \Big/ \left\{\frac{36n^2-12n+1-1}{6}-1\right\} \\
&= \frac{6n^2+10n+3}{6n^2-2n-1} \qquad (1)
\end{aligned}
$$

As mentioned in section 2, the first and second numbers of an ATP pair start from integer 5. As a result, the ATP Group (ATPG0) would be equal to 3 if n = 0, i.e., for the ATP pairs (5, 7), (11, 13), and (17, 19); and if the range is smaller than $(6n + 5)^2 = 25$ when n = 0.

As a result, ATPGn is presented as (ATPGn-1) (ATPGn%) and expanded to (ATPG0) (ATPG1%) (ATPG2%) (ATPG3%) … (ATPGn%), which is calculated as follows:

$$ATPGn = 3\prod_{i=1}^{n}\frac{6i^2+10i+3}{6i^2-2i-1} \qquad (2)$$

where the initial value is 3 when n = 0, i.e., $(6n + 5)^2 = 25$ when n = 0.

By employing the above equation, we obtain ATPG0 = 3, ATPG1 = ATPG0 · ATPG1% = $3\left(\frac{19}{3}\right)$ = 19, ATPG2 = ATPG1 · ATPG2% = $3\left(\frac{19}{3}\right)\left(\frac{47}{19}\right)$ = 47, and so on.

In addition, we further measure the equation TPEn = (ATPGn) (TPRn), where TPEn is the estimated number of twin prime pairs accumulated in the range less than $(6n + 5)^2$ for n = 1 to ∞, ATPGn is the number of ATP pairs when the integer is less than $(6n + 5)^2$ for n = 1 to ∞, and TPRn is the probability of ATP pairs being twin prime pairs within the range of $(6n + 5)^2$ for n = 1 to ∞. As a result, we can measure TPEn, i.e., the estimated number of twin prime pairs accumulated from the range $(6n + 5)^2$ when n = 0 (initial value = 3) to the range $(6n + 5)^2$ when n = ∞.

In this study, we define TPEn as the estimated number of twin prime pairs accumulated within a range less than $(6n + 5)^2$ for n = 1 to ∞, ATPGn as the number of ATP pairs when the integer is less than $(6n + 5)^2$ for n = 1 to ∞, and TPRn as the probability of ATP pairs being twin prime pairs within the range of $(6n + 5)^2$ for n = 1 to ∞. Then, by measuring the equation TPEn = (ATPGn) (TPRn), we can measure TPEn within the range of $(6n + 5)^2$ when n = 1[1] to $(6n + 5)^2$ when n = ∞. In other words, we can measure whether twin prime pairs would increase unlimitedly as the incremental range $(6n + 5)^2$ increases from 1 to ∞. The topic is discussed in the following section.

4. Probability of ATP Pairs Being Twin Prime Pairs

ATPn includes a pair of natural numbers ($ATPn^a$, $ATPn^b$). As shown in Figure 1, when selecting any natural number x for ATPn (e.g., $ATPn^a$ = 5 when n = 1), two ATP pairs out of five would not be twin prime pairs; similarly, when selecting any natural number x for ATPn (e.g., $ATPn^b$ = 7 when n = 1), two ATP pairs out of seven would not be twin prime pairs. In other words, the probability of excluding twin prime pairs from ATP pairs is 2/x. For example, the probability of excluding twin prime pairs from ATP pairs is 2/5 when x is 5 (i.e., $ATP1^a$ = 5) and 2/7 when x is 7 (i.e.,$ATP1^b$ = 7).

In Figure 1, the numbers in black boxes are excluded as prime numbers because they are divisible by 5 or 7. When the first or second number in an ATP pair (6n – 1, 6n + 1) is not a prime number, the pair is not considered as a twin prime pair. We also find that excluding twin prime pairs from ATP pairs would be regular, i.e., the probability of excluding twin prime pairs from ATP pairs would be 2/x.

According to the screening approach proposed by the sieve of Eratosthenes, some ATP pairs would be excluded from twin prime pairs because the multiples of prime numbers that are smaller than $\sqrt{N}$ are shown in the

ATP Pairs	A	b		ATP Pairs	a	b	
1	$\boxed{5}$	7		1	5	$\boxed{7}$	
2	11	13		2	11	13	
3	17	19		3	17	19	
4	23	$\boxed{25}$	5 pairs	4	23	25	
5	29	31		5	29	31	7 pairs
6	$\boxed{35}$	37		6	$\boxed{35}$	37	
7	41	43		7	41	43	
8	47	49		8	47	$\boxed{49}$	
9	53	$\boxed{55}$	5 pairs	9	53	55	
10	59	61		10	59	61	
11	$\boxed{65}$	67		11	65	67	
12	71	73		12	71	73	7 pairs
13	77	79		13	$\boxed{77}$	79	
14	83	$\boxed{85}$	5 pairs	14	83	85	
15	89	91		15	89	$\boxed{91}$	
16	$\boxed{95}$	97		16	95	97	
17	101	103		17	101	103	
18	107	109		18	107	109	
19	113	$\boxed{115}$	5 pairs	19	113	115	7 pairs
20	119	121		20	$\boxed{119}$	121	
21	$\boxed{125}$	127		21	125	127	
22	131	133		22	131	$\boxed{133}$	
23	137	139		23	137	139	
24	143	$\boxed{145}$	5 pairs	24	143	145	
25	149	151		25	149	151	
26	$\boxed{155}$	157		26	155	157	7 pairs
27	161	163		27	$\boxed{161}$	163	
28	167	169		28	167	169	
29	173	$\boxed{175}$	5 pairs	29	173	$\boxed{175}$	
30	179	181		30	179	181	
31	$\boxed{185}$	187		31	185	187	
32	191	193		32	191	193	
33	197	199		33	197	199	7 pairs
34	203	$\boxed{205}$	5 pairs	34	$\boxed{203}$	205	
35	209	211		35	209	211	
36	$\boxed{215}$	217		36	215	$\boxed{217}$	

Figure 1. Excluding twin prime pairs from ATP pairs for prime numbers 5 and 7.

[1] The initial value = 3 for TPEn when n = 0 because three twin prime pairs, (3, 5), (11, 13), and (17, 19), are less than 25 (i.e., $(6n + 5)^2$ when n = 0).

first or second number of ATP pairs.

Some of the first and second numbers in the ATP pair are composite numbers, which would not be considered as twin prime pairs. For example, the numbers 35, 175, and 245 in an ATP pair would be excluded because these numbers are composite numbers.

Thus, we prove that at least one twin prime pair exists within the incremental range from $(6n + 5)^2$ to $[6(n + 1) + 5]^2$ for n = 1, 2, 3, … ∞, when the initial value = 3 as n = 0.[2] In addition, we argue that our proof would be more persuasive if we can prove that at least one twin prime pair exists within the incremental range. In addition, we document that our estimation would be underestimated, as explained below.

While setting $\sqrt{N}$ = 6n – 1 or 6n + 1, N would be a prime number close to $(6n + 5)^2$, i.e., $[6(n + 1) - 1]^2 = N$, which could not be divided by any prime number smaller than $\sqrt{N}$, such as 6n + 1,because 6n + 1 is smaller than 6n + 5, i.e., $6n + 1 < 6n + 5 = \sqrt{N}$. As a result, ATPG1 (i.e., ATPGn when n = 1) could not be divided by a prime number smaller than $\sqrt{121}$ (i.e., 5 and 7), ATPG2 (i.e., ATPGn when n = 2) could not be divided by a prime number smaller than $\sqrt{289}$ (i.e., 5, 7, 11, and 13), ATPG3 (i.e., ATPGn when n = 3) could not be divided by a prime number smaller than $\sqrt{529}$ (i.e., 5, 7, 11, 13, 17, and 19), ATPG4 (i.e., ATPGn when n = 4) could not be divided by a prime number smaller than $\sqrt{841}$ (i.e., 5, 7, 11, 13, 17, 19, 23); ATPG5 (i.e., ATPGn when n = 5) could not be divided by a prime number smaller than $\sqrt{1225}$ (i.e., 5, 7, 11, 13, 17, 19, 23, 29, 31), and so on. The above explanation should benefit the measurement of the probability of Cases I-IV. In addition, the probability of Case I would be regarded as a conservative probability.

In this study, the probability of Case I, i.e., $ATPn^a$ and $ATPn^b$ are all prime numbers, is $TPRn^{ab}$ (the ratio of twin prime pairs to ATP pairs; twin prime ratio), which is measured as follows:

$$TPRn^{ab} = \left(1 - \frac{2}{ATPn^a}\right)\left(1 - \frac{2}{ATPn^b}\right) \tag{3}$$

We also find that excluding twin prime pairs from ATP pairs would be regular, i.e., the probability of excluding twin prime pairs from ATP pairs is 2/x, where x is the first or second number in the ATP pair ($ATPn^a$, $ATPn^b$).

However, we argue that four cases exist for the ATP pair ($ATPn^a$, $ATPn^b$): Case I (prime number, prime number), Case II (prime number, composite number), Case III (composite number, prime number), and Case IV (composite number, composite number). We can measure various probabilities for Cases II-IV, as shown below.

$TPRn^a = \left(1 - \frac{2}{ATPn^a}\right)$ for Case II: $ATPn^a$ is a prime number, whereas $ATPn^b$ is a composite number.

$TPRn^b = \left(1 - \frac{2}{ATPn^b}\right)$ for Case III: $ATPn^a$ is a composite number, whereas $ATPn^b$ is a prime number.

$TPRn^0 = 1$ for Case IV: $ATPn^a$ and $ATPn^b$ are composite numbers.

In addition, we can compare the probabilities for these cases.

$$\left(1 - \frac{2}{ATPn^a}\right)\left(1 - \frac{2}{ATPn^b}\right) < 1 - \frac{2}{ATPn^a} < 1 - \frac{2}{ATPn^b} < 1$$

That is, $TPRn^{ab} < TPRn^a < TPRn^b < TPRn^0$.

We employ the probability $TPRn^{ab}$ rather than $TPRn^a$, $TPRn^b$, and $TPRn^0$ to measure the existence of numerous twin prime pairs. Thus, our estimation is performed conservatively.

$$\begin{aligned} TPRn &= \prod_{i=1}^{n}\left(1 - \frac{2}{ATPi^a}\right)\left(1 - \frac{2}{ATPi^b}\right) \\ &= \prod_{i=1}^{n}\left(1 - \frac{2}{6i-1}\right)\left(1 - \frac{2}{6i+1}\right) \\ &= \prod_{i=1}^{n}\left(\frac{6i-1-2}{6i-1}\right)\left(\frac{6i+1-2}{6i+1}\right) \\ &= \prod_{i=1}^{n}\frac{(6i-3)(6i-1)}{(6i-1)(6i+1)} \\ &= \prod_{i=1}^{n}\frac{(6i-3)}{(6i+1)} \end{aligned} \tag{4}$$

[2] ATPG is equal to 3, including (5, 7), (11, 13), and (17, 19), for integers smaller than 5^2 (i.e., $(6n + 5)^2 = 25$ when n = 0). In addition, twin prime pairs are equal to 3 as well because (5, 7), (11, 13), and (17, 19) are all twin prime pairs.

5. Evidence of the Existence of Infinite Twin Prime Pairs

Twin prime pairs are conservatively estimated as follows: the number of ATP groups (ATPGn) is multiplied by the probability of twin prime pairs in ATP pairs (TRPn).

Thus, TPEn = (ATPGn) (TPRn), where TPEn is the estimated number of twin prime pairs accumulated within the range less than $(6n + 5)^2$ for n = 1 to ∞, ATPGn is the number of ATP pairs when the integer is less than $(6n + 5)^2$ for n = 1 to ∞, and TPRn is the probability of ATP pairs being twin prime pairs within the range of $(6n + 5)^2$ for n = 1 to ∞. We then employ $TPRn^{ab}$ instead of $TPRn^{a}$, $TPRn^{b}$, and $TPRn^{0}$ to measure the above equation. Similarly, the number of estimated twin prime pairs is conservative.

As $ATPGn = 3\prod_{i=1}^{n}\frac{6i^2+10i+3}{6i^2-2i-1}$ and $TPRn^{ab} = \prod_{i=1}^{n}\frac{(6i-3)}{(6i+1)}$,

$$\begin{aligned} TPEn &= 3\prod_{i=1}^{n}\left(\frac{6i^2+10i+3}{6i^2-2i-1}\right)\left(\frac{6i-3}{6i+1}\right)\\ &= 3\prod_{i=1}^{n}\left(\frac{36i^3+60i^2+18i-18i^2-30i-9}{36i^3-12i^2-6i+6i^2-2i-1}\right)\\ &= 3\cdot\prod_{i=1}^{n}\left(\frac{36i^3+42i^2-12i-9}{36i^3-6i^2-8i-1}\right) \end{aligned} \tag{5}$$

To prove twin prime conjecture, we simplify the above math equation as

$$\begin{aligned} &= 3\prod_{i=1}^{n}\left(\frac{36i^3+42i^2-12i-9}{36i^3-6i^2-8i-1}\right)\\ &= 3\prod_{i=1}^{n}\left[\frac{(36i^3-6i^2-8i-1)+(48i^2-4i-8)}{36i^3-6i^2-8i-1}\right]\\ &= 3\prod_{i=1}^{n}\left[1+\frac{(48i^2-4i-8)}{36i^3-6i^2-8i-1}\right]\\ &= 3\prod_{i=1}^{n}\left[1+\frac{\left(36i^2-6i-8-\frac{1}{i}\right)}{36i^3-6i^2-8i-1}+\frac{\left(12i^2+2i+\frac{1}{i}\right)}{36i^3-6i^2-8i-1}\right]\\ &= 3\prod_{i=1}^{n}\left[1+\frac{\left(36i^2-6i-8-\frac{1}{i}\right)}{i\left(36i^2-6i-8-\frac{1}{i}\right)}+\frac{\left(12i^2+2i+\frac{1}{i}\right)}{36i^3-6i^2-8i-1}\right]\\ &= 3\prod_{i=1}^{n}\left[1+\frac{1}{i}+\frac{\left(12i^2+2i+\frac{1}{i}\right)}{36i^3-6i^2-8i-1}\right]\\ &> 3\prod_{i=1}^{n}\left[1+\frac{1}{i}\right] \quad \because i\ge 1,\quad \frac{\left(12i^2+2i+\frac{1}{i}\right)}{36i^3-6i^2-8i-1}>0 \end{aligned} \tag{6}$$

The twin prime pairs are greater than $3\prod_{i=1}^{n}\left[1+\frac{1}{i}\right]$ because $\frac{\left(12i^2+2i+\frac{1}{i}\right)}{36i^3-6i^2-8i-1}$ is not negative. After simplifying $3\prod_{i=1}^{n}\left[1+\frac{1}{i}\right]$ we obtain $3\prod_{i=1}^{n}\left[1+\frac{1}{i}\right]$, which is equal to 3n + 3.

$$\begin{aligned} &3\prod_{i=1}^{n}\left[1+\frac{1}{i}\right]=3\prod_{i=1}^{n}\left[\frac{i+1}{i}\right]=3\left(\frac{\cancel{2}}{1}\right)\left(\frac{\cancel{3}}{\cancel{2}}\right)\cdots\left(\frac{\cancel{n}}{\cancel{n-1}}\right)\left(\frac{n+1}{\cancel{n}}\right)\\ &=3(n+1)=3n+3 \end{aligned} \tag{7}$$

After differentiating the above equation, we determine that this conservative estimation is an incremental function.

$$\frac{d}{dn}(3n+3)=3 \tag{8}$$

The above inference indicates that three twin prime pairs increase as n is increased by 1.

Thus, we prove the twin prime conjecture proposed by de Polignac in 1849.

In addition, we measure TPEn (i.e., TPEn = (ATPGn) (TPRn) while setting $TPRn = TPRn^{ab}$), TPAn (the actual twin prime pairs), and TPEn-S (the conservative estimation for measuring TPE due to the exclusion of non-negative terms, as well as setting $TRRn = TPRn^{ab}$) for n = 1 to 50 (Table 2). We observe that TPEn is less than TPAn and that TPEn-S is far less than TPAn. Ne-

Table 2. Measurements for TPEn, TPAn, and TPE-S

n	ATPGn		TPRn	TPEn	TPAn	TPEn-S
	Incremental range	The number of ATP pairs	Conservation			Conservative & simplified
	$(6n+5)^2$	$\left\{\frac{[6n+5]^2-1}{6}-1\right\}$	(1) The conservative probabilities of $TPRn^{ab}$: $\prod_{i=1}^{n}\left(1-\frac{2}{ATPi^a}\right)\left(1-\frac{2}{ATPi^b}\right)$	TPEn	Actual TP numbers	1. The conservative probabilities of $TPRn^{ab}$: $\prod_{i=1}^{n}\left(1-\frac{2}{ATPi^a}\right)\left(1-\frac{2}{ATPi^b}\right)$ 2. Excluding the non-negative terms
1	121	19	42.86%	8	9	6
2	289	47	29.67%	14	18	9
3	529	87	23.42%	20	24	12
4	841	139	19.68%	27	32	15
5	1,225	203	17.14%	35	40	18
6	1,681	279	15.28%	43	52	21
7	2,209	367	13.86%	51	66	24
8	2,809	467	12.73%	59	79	27
9	3,481	579	11.81%	68	92	30
10	4,225	703	11.03%	78	109	33
11	5,041	839	10.37%	87	127	36
12	5,929	987	9.80%	97	142	39
13	6,889	1,147	9.31%	107	159	42
14	7,921	1,319	8.87%	117	173	45
15	9,025	1,503	8.48%	127	190	48
16	10,201	1,699	8.13%	138	209	51
17	11,449	1,907	7.81%	149	226	54
18	12,769	2,127	7.53%	160	243	57
19	14,161	2,359	7.27%	171	262	60
20	15,625	2,603	7.03%	183	277	63
21	17,161	2,859	6.80%	195	297	66
22	18,769	3,127	6.60%	206	324	69
23	20,449	3,407	6.41%	218	346	72
24	22,201	3,699	6.23%	231	375	75
25	24,025	4,003	6.07%	243	401	78
26	25,921	4,319	5.91%	255	417	81
27	27,889	4,647	5.77%	268	443	84
28	29,929	4,987	5.63%	281	466	87
29	32,041	5,339	5.50%	294	491	90
30	34,225	5,703	5.38%	307	526	93
31	36,481	6,079	5.27%	320	553	96
32	38,809	6,467	5.16%	334	580	99
33	41,209	6,867	5.05%	347	601	102
34	43,681	7,279	4.95%	361	629	105
35	46,225	7,703	4.86%	374	654	108
36	48,841	8,139	4.77%	388	684	111
37	51,529	8,587	4.69%	402	722	114
38	54,289	9,047	4.60%	417	747	117
39	57,121	9,519	4.53%	431	779	120
40	60,025	10,003	4.45%	445	810	123
41	63,001	10,499	4.38%	460	834	126

Table 2. Continued

n	ATPGn		TPRn	TPEn	TPAn	TPEn-S
	Incremental range	The number of ATP pairs	Conservation			Conservative & simplified
	$(6n+5)^2$	$\left\{\frac{[6n+5]^2-1}{6}-1\right\}$	(1) The conservative probabilities of $TPRn^{ab}$: $\prod_{i=1}^{n}\left(1-\frac{2}{ATPi^a}\right)\left(1-\frac{2}{ATPi^b}\right)$	TPEn	Actual TP numbers	1. The conservative probabilities of $TPRn^{ab}$: $\prod_{i=1}^{n}\left(1-\frac{2}{ATPi^a}\right)\left(1-\frac{2}{ATPi^b}\right)$ 2. Excluding the non-negative terms
42	66,049	11,007	4.31%	474	867	129
43	69,169	11,527	4.24%	489	894	132
44	72,361	12,059	4.18%	504	937	135
45	75,625	12,603	4.12%	519	964	138
46	78,961	13,159	4.06%	534	994	141
47	82,369	13,727	4.00%	549	1,036	144
48	85,849	14,307	3.94%	564	1,074	147
49	89,401	14,899	3.89%	580	1,108	150
50	93,025	15,503	3.84%	595	1,147	153

Note: TPEn = (ATPGn) (TPRn) while setting TPRn = $TPRn^{ab}$ for n = 1 to 50.
TPAn is the actual twin prime pair for n = 1 to 50.
TPEn-S is the conservative estimation for measuring TPE due to the exclusion of non-negative terms, as well as setting TRRn = $TPRn^{ab}$ for n = 1 to 50.
In addition, TPAn (actual TP numbers) might have the following issues. (1) The probabilities measured by $TPRn^a$, $TPRn^b$, $TPRn^0$, or $TPRn^{ab}$ might have depended on various cases shown in ($ATPn^a$, $ATPn^b$), and (2) non-negative terms were excluded.

vertheless, we confirm that numerous twin prime pairs exist, as indicated by the results for TPEn-S and TPEn.

We then plot the trend for TPEn, TPAn, and TPEn-S. Setting $(6n+5)^2$ as the incremental range for measuring the number of twin prime pairs, we observe increasing convex curves for TPEn and TPAn and an increasing linear curve for TPn-S because TPE-S = 3n + 3. Moreover, TPAn is significantly higher than TPEn-S and TPEn. While n approaches ∞, we confirm numerous twin prime pairs for TPEn-S. However, the number of twin prime pairs measured by TPEn-S is far less than that measured by TPEn. Similarly, the number of twin prime pairs measured by TPEn is far less than that measured by TPAn, thus confirming the existence of numerous twin prime pairs.

The above estimation is regarded as a conservative estimation, which is achieved as follows. (1) We employ $TPRn^{ab}$ instead of $TPRn^a$, $TPRn^b$, and $TPRn^0$ to measure TPRn-S. (2) We simplify the mathematical presentation by excluding the non-negative term $\frac{\left(12i^2+2i+\frac{1}{i}\right)}{36i^3-6i^2-8i-1}$. Nonetheless, we prove the existence of numerous prime numbers p for each natural number k by making p + 2k as a prime number for the case of k = 1.

If we employ $TPRn^{ab}$, $TPRn^a$, $TPRn^b$, or $TPRn^0$ depending on the actual condition and include a non-negative term in the mathematical equation, the number of twin prime pairs would increase relative to the conservative estimation above. Moreover, the estimated results would be close to the actual ones; how this can be achieved may be explored in further.

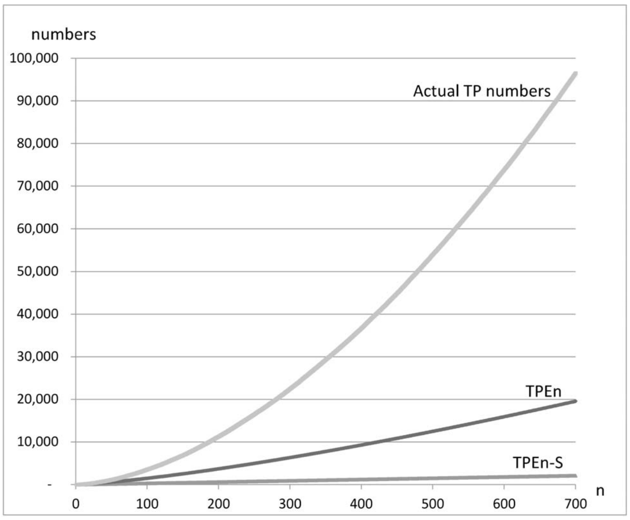

Figure 2. Trend for TPEn, TPAn, and TPEn-S. Note: TPAn: actual twin prime numbers considering the two conditions below.
TPEn: conservative twin numbers measured by the equation ATPGn · TPRn = TPEn, where TPRn is set as $\prod_{i=1}^{n}\left(1-\frac{2}{ATPi^a}\right)\left(1-\frac{2}{ATPi^b}\right)$.
TPEn-S: conservative and simplified measurement for twin primes as shown above. (1) TPRn is set as $\prod_{i=1}^{n}\left(1-\frac{2}{ATPi^a}\right)\left(1-\frac{2}{ATPi^b}\right)$. (2) Non-negative terms are excluded.

6. Conclusion

We extend the sieve of Eratosthenes to calculate the existence of numerous twin prime pairs. We perform a conservative estimation to make the results reliable and convincing.

Although estimation is performed conservatively, we prove that at least three additional twin prime pairs increase when $(6n + 5)^2$ is set as the range in estimation as n is increased by 1 for n = 1, 2, … ∞.

We then confirm that additional twin prime pairs increase as n is increased. Moreover, twin prime pairs could increase to infinity because n could be increased from 1 to ∞. Thus, Hilbert's eighth unsolved problem (twin prime conjecture) is resolved in this study.

Twin prime conjecture is solved on the basis of two main conditions: (1) deleting the non-twin prime pairs in ATP pairs would be regular (i.e., 2/x shown in context) and (2) setting the incremental range from $(6n + 5)^2$ to $[6(n + 1) + 5]^2$ for n = 1, 2, 3, … ∞ (i.e., the increased incremental range would also be regarded as regular). We argue that twin prime conjecture is solved in this study because these conditions are considered deliberately.

Hence, the estimation of whether there are infinite numbers of Twin Prim pairs is rather conservative and even simplified, the more accurate estimation would be taken into account for further studies.

This study contributes to the literature in the following ways. First, we denote the range of $(6n + 5)^2$ as our extended scope. We derive at least three twin prime pair samples as n is increased by 1. Without finding the incremental range of $(6n + 5)^2$ as n is increased by 1, we cannot infer or even find the approximate solution for twin prime conjecture. In addition, to the best of our knowledge and because of the limited knowledge for the proof of a mathematical approach, we believe that this study is the first to employ a scientific method to infer twin prime conjecture by employing the incremental range of $(6n + 5)^2$. We ultimately confirm twin prime conjecture reasonably. Second, the prime number theorem, which states that $x/\log x$ is a good approximation to $\pi(x)$, is also a well-recognized discovery. We propose a close estimation formula rather than a precise one. On the basis of this formula, we provide an approximate solution for twin prime conjecture.

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