Numerical Treatment On SEIR Problem With Temporal Fractal Fractional Derivatives

Jing Ye* and Shengyi Zhou

Sichuan Vocational College of Chemical Technology, Luzhou, Sichuan, 646300, China

Corresponding author. E-mail:zhaoming198155@scpolicec.edu.cn

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In this work, a well-known epidemic SEIR model is considered with the fractal-fractional operator in the frame of the Atangana-Baleanu derivative. Moreover, Using the theorems of Schauders fixed point and Banach fixed, existence theory is practiced to guarantee that there are solutions to the model. Approximate solutions to the problem are presented using the Atangana-Toufik scheme. Also, 2D graphs of solutions for different fractional orders are shown. Along with chaotic behavior of results for each case are investigated.

Keywords: Mittag-Leffler kernel; numerical method; fractional SEIR epidemic model.

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1. Introduction

Mathematical modeling performs a pivotal function in different areas. Most the phenomena can be described by a suitable model which derives as a dynamical system or single differential equation. Among the models, epidemic models play a significant role for mathematicians and scientists. We can mention some of the epidemic models in literature, such as the HIV-1 infection model [1], hepatitis-B virus [2, 3], cancer model [4], an so on [5–9].

The SEIR model is a development of the classical SIR model. In most diseases, after the initial infection, the host remains in a latent stage before becoming infectious, and this period may not be negligible compared to the infectious period. Because of this, it makes sense to add a new compartment to the epidemiological model. The population is divided into susceptible, exposed, infectious, and recovered (SEIR), with treatment policies.

Designing the model of SEIR epidemic dilemmas has matured into a latent branch of experimentation to study multiple attributes of SEIR epidemic problems. Scientists are engrossed in the investigation of the role of this problem using various mathematical techniques and standards. Various non-integer order operators were adopted for the examination of problems [10–19].

Many natural phenomena can be characterized exactly with the fractional calculation [20–22]. To characterize the transference dynamics of bedbugs regulate a deterministic, compartmental system. The community is consistently fraternized and iterates the demography of a common emerging nation, as it studies exponentially growing dynamics. The whole crowd (N) is classified into four categories S, E, I and R represented as susceptible, exposed, infectious, and recovered crowd in the order already mentioned to illustrate the system equations [23]:

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= B - \vartheta_1 S(t) I(t) - \mu S(t) \\
\frac{\mathrm{d}E}{\mathrm{d}t} &= \vartheta_1 S(t) I(t) - (\sigma + \mu + \delta_1) E(t) \\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \delta_1 E(t) - (\gamma_1 + \mu) I(t) \\
\frac{\mathrm{d}R}{\mathrm{d}t} &= \gamma_1 I(t) + \sigma E(t) - \mu R(t)
\end{aligned} \tag{1}$$

Where B is the birth and natural death rate (we assume they are equal), $\vartheta 1$ is the disease transmission rate, σ is the incubation rate, μ the recovery rate (without treatment), and γ_1 the treatment rate.

The Atangana-Baleanu derivative is a nonlocal fractional derivative with a nonsingular kernel, which is connected with a variety of applications. By this motivation, the fractionalorder extension of the above system has been first analyzed in [24, 25] showing the sensible having two phases decrease treatment of contamination of diseases. Nonlocal differential and integral operators with fractional order and fractal dimension have been recently introduced and appear to be powerful mathematical tools to model complex real-world problems that could not be modeled with classical and nonlocal differential and integral operators with single order [26–32].

The importance of the epidemic models in human life and analyzing them with different instruments, especially with mathematical modeling, and different operators, motivates us to develop the SEIR model with the fractal fractional operator in the Atangana-Baleanu fractional derivative. It is well-known that, when the order of fractional derivative goes to integer one, fractional derivative tends to classical derivative. Therefore, the results of this paper contain the solutions of the SEIR model with integer derivatives, when we assume the fractional order to be one. We consider the Eq. (1) using fractal-fractional operator in Atangana-Baleanu frame:

$$\begin{aligned}
&{}_{0}^{FFM}\mathcal{D}_t^{\delta,\vartheta} S(t) = B - \vartheta_1 S(t)I(t) - \mu S(t)\\
&{}_{0}^{FFM}\mathcal{D}_t^{\delta,\vartheta} E(t) = \vartheta_1 S(t)I(t) - (\sigma + \mu + \delta_1)\, I(t),\\
&{}_{0}^{FFM}\mathcal{D}_t^{\delta,\vartheta} I(t) = \delta_1 I(t) - (\gamma_1 + \mu)\, I(t)\\
&{}_{0}^{FFM}\mathcal{D}_t^{\delta,\vartheta} R(t) = \gamma_1 I(t) + \sigma E(t) - \mu R(t)
\end{aligned} \tag{2}$$

Using $S(0) = S_0, E(0) = E_0, I(0) = I_0$ and $R(0) = R_0(t)$

In this work, we investigate the existence and stability, and Atangana-Toufik approach for problem Eq. (2). Different investigations have been done for the SEIR model. LAD approach has been used to solve this problem in [25]. Also, in [33] finite difference scheme is applied to this problem. Moreover, modeling of this problem was investigated in [34]. In [35] the transform method has been employed for this problem. Indeed, an analytical solution to this problem can be found in [36]. Moreover, asymptotic approximations have been used for this problem in [37]. Some other applications can be found in [38–40].

Definition 1.1.[41] Suppose $u \in C(a,b)$, then:

$$\begin{aligned}
{}^{FFM}\mathcal{D}_t^{\delta,\vartheta}(u(t)) &= \frac{\mathcal{N}(\delta)}{1-\delta}\frac{d}{dt^{\vartheta}}\\
&\times \int_a^t u(z) E_\delta\left[\frac{-\delta}{1-\delta}(t-z)^{\delta}\right] \mathrm{d}z,\\
0 < \delta, \vartheta &\le 1,
\end{aligned} \tag{3}$$

and $\mathcal{N}(\delta)$ is the normalization function which $\mathcal{N}(0) = \mathcal{N}(1) = 1$.

Definition 1.2. [41] Consider continuous function $u(t)$ on (a,b), then

$$\begin{aligned}
{}^{\mathrm{FFM}}\mathcal{I}_t^{\delta,\vartheta}(u(t)) &= \frac{\vartheta\delta}{\mathcal{N}(\delta)\Gamma(\delta)}\\
&\times \int_0^t z^{\vartheta-1} u(z)(t-z)^{\delta-1} dz\\
&+ \frac{\vartheta(1-\delta)t^{\vartheta-1}}{\mathcal{N}(\delta)} u(t),\\
0 < \delta, \vartheta &\le 1
\end{aligned} \tag{4}$$

is called the fractal-fractional integral of u

The plan of the paper is as follows.

In Section 2, we give the existence and uniqueness results of the considered model. Section 3, deals with the Ulam-Hyres stability analysis of the model. Numerical results and simulations are discussed in Section 4. Finally, in Section 5, we draw our conclusions.

2. Existence and uniqueness results

Now, we show the being of a solution for system Eq. (2) using fixed point theorem. Consider system Eq. (2) as

$$\left\{\begin{aligned}
&{}_{0}^{ABR}D_t^{\delta} S(t) = \vartheta t^{\vartheta-1}\mathcal{Q}(t,S,E,I,R),\\
&{}_{0}^{ABR}D_t^{\delta} E(t) = \vartheta t^{\vartheta-1}\mathcal{W}(t,S,E,I,R),\\
&{}_{0}^{ABR}D_t^{\delta} I(t) = \vartheta t^{\vartheta-1}\mathcal{E}(t,S,E,I,R),\\
&{}_{0}^{ABR}D_t^{\delta} R(t) = \vartheta t^{\vartheta-1}\mathcal{Z}(t,S,E,I,R),
\end{aligned}\right. \tag{5}$$

With

$$\left\{\begin{aligned}
&\mathcal{Q}(t,S,E,I,R) = B - \vartheta_1 S(t)I(t) - \mu S(t)\\
&\mathcal{W}(t,S,E,I,R) = \vartheta_1 S(t)I(t) - (\sigma + \mu + \delta_1)\, I(t)\\
&\mathcal{M}(t,S,E,I,R) = \delta_1 E(t) - (\gamma_1 + \mu)\, I(t)\\
&\mathcal{Z}(t,S,E,I,R) = \gamma_1 I(t) + \sigma E(t) - \mu R(t)
\end{aligned}\right. \tag{6}$$

Now, Eq. (5) can be written as:

$$\left\{\begin{aligned}
&{}_{0}^{ABR}D_t^{\delta}\Psi(t) = \vartheta t^{\vartheta-1}\Lambda(t,\Psi(t))\\
&\Psi(0) = \Psi_0.
\end{aligned}\right. \tag{7}$$

Substituting ${}_{0}^{ABR}Dt^{\delta}$ by ${}_{0}^{ABC}D_t^{\delta\vartheta}$ and employing fractional integral, we obtain

$$\begin{aligned}
\Psi(t) &= \Psi(0) + \frac{\vartheta t^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)}\Lambda(t,\Psi(t))\\
&+ \frac{\delta\vartheta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^t \omega^{\vartheta-1}(t-\omega)^{\vartheta-1}\Lambda(\omega,\Psi(\omega))d\omega,
\end{aligned} \tag{8}$$

Where

$$\Psi(t)=\left\{\begin{array}{l} S(t), \\ E(t), \\ I(t), \\ R(t), \end{array}\right., \Psi(0)=\left\{\begin{array}{l} S(0), \\ E(0), \\ I(0), \\ R(0), \end{array}\right.$$

$$\Lambda(t,\Psi(t))=\left\{\begin{array}{l} \mathcal{Q}(t,S,E,I,R) \\ \mathcal{W}(t,S,E,I,R) \\ \mathcal{M}(t,S,E,I,R) \\ \mathcal{Z}(t,S,E,I,R) \end{array}\right.$$

Due to existence hypothesis, a Banach space $\mathcal{B}=\mathcal{Y}\times\mathcal{Y}\times\mathcal{Y}\times\mathcal{Y}$ is assigned, where $\mathcal{Y}=\mathbb{C}[0,\mathbb{T}]$ using

$$\|\Psi\|=\max_{t\in[0,\mathbb{T}]}|S(t)+E(t)+I(t)+R(t)|.$$

Assign $\mathcal{L}:\mathcal{B}\rightarrow\mathcal{B}$ as:

$$\begin{aligned} \mathcal{L}(\Psi)(t)=\Psi(0)+\frac{\vartheta t^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)}\Lambda(t,\Psi(t)) \\ +\frac{\delta\vartheta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^t(t-\omega)^{\vartheta-1}\Lambda(\omega,\Psi(\omega))d\omega \end{aligned}\tag{9}$$

Imposing Lipschitz condition $\Lambda(t,\Psi(t))$ as:

- For $\Psi\in\mathcal{B}$, there exist constants $\mathcal{G}_\Lambda>0$ and M_Λ such that

$$|\Lambda(t,\Psi(t))-\Lambda(t,\bar{\Psi}(t))|\leq\mathcal{L}_\Lambda|\Psi(t)-\bar{\Psi}(t)|\tag{10}$$

- For each $\Psi,\bar{\Psi}\in\mathcal{B}$, there exist a constant $\mathcal{L}_\Lambda>0$ such that

$$|\Lambda(t,\Psi(t))-\Lambda(t,\bar{\Psi}(t))|\leq\mathcal{L}_\Lambda|\Psi(t)-\bar{\Psi}(t)|\tag{11}$$

Now, we are ready to consider the existence of solution for the given system. The existence of a solution for a model is the first condition to consider the approximate solutions or stability of the problem.

Theorem 2.1. Consider relation Eq. (10) works. Suppose $\Lambda:[0,\mathbb{T}]\times\mathcal{B}\rightarrow\mathbb{R}$ be a continuous. So, the presented system has a solution.

Proof. We confirm $\mathcal{L}$ expressed via Eq. (9) is continuous. Because Λ is continuous, so, $\mathcal{L}$ is continuous.

Consider $\mathbb{H}=\{\Psi\in\mathcal{B}:\|\Psi\|\leq\mathcal{R},\mathcal{R}>0\}$. For each $\Psi\in\mathcal{B}$, we own

$$\begin{aligned} \|\mathcal{L}(\Psi)\|=&\max_{t\in[0,\mathbb{T}]}\mid\Psi(0)+\frac{\vartheta t^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)}\Lambda(\omega,\Psi(\omega)) \\ &+\frac{\delta\vartheta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^t\omega^{\vartheta-1}(t-\omega)^{\vartheta-1}\Lambda(\omega,\Psi(\omega))d\omega\mid \\ &\leq\Psi(0)+\frac{\vartheta\mathbb{T}^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)}(\mathcal{G}\|\Psi\|+M_\Lambda) \\ &+\max_{t\in[0,\mathbb{T}]}\frac{\delta\vartheta(\delta)\Gamma(\delta)}{\mathcal{N}}\int_0^t\omega^{\theta-1}|\Lambda(\rightarrow,\Psi(\omega))|d\omega \\ &+\frac{\delta\vartheta}{\mathcal{N}(\delta)\Gamma(\delta)}(\mathcal{G}\|\Psi\|+M_\Lambda)\mathbb{T}^{\delta+\vartheta-1}\mathcal{H}(\delta,\vartheta)\leq\mathcal{R}. \\ &\leq\Psi(0)+\frac{\vartheta\mathbb{T}^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)}(\mathcal{G}\|\Psi\|+M_\Lambda) \end{aligned}$$

Then $\mathcal{L}$ is limited, which $\mathcal{H}(\delta,\vartheta)$ indicates function of beta. For equicontinuity $\mathcal{L}$, we consider $t_1\leq t_2\leq\mathbb{I}$. Thus, take

$$\begin{aligned} &|\mathcal{L}(\Psi)(t_2)-\mathcal{L}(\Psi)(t_1)|=\mid\frac{\vartheta t_2^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)}\Lambda(t_2,\Psi(t_2)) \\ &+\frac{\delta\vartheta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^{t_2}\omega^{\theta-1}(t_2-\omega)^{\vartheta-1}\Lambda(\omega,\Psi(\omega))d\omega \\ &-\frac{\vartheta t_1^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)}\Lambda(t_1,\Psi(t_2)) \\ &+M_\Lambda)+\frac{\delta\vartheta}{\mathcal{N}(\delta)\Gamma(\delta)}(\mathcal{G}|\Psi(t)|+M_\Lambda)t_2^{\delta+\theta-1}\mathcal{H}(\delta,\vartheta) \\ &\leq\frac{\vartheta t_2^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)}(\mathcal{G}_\Lambda|\Psi(k)|+M_\Lambda) \\ &-\frac{\delta\vartheta}{\mathcal{N}(\delta)\Gamma(\delta)}(\mathcal{G}|\Psi(t)|+M_\Lambda)t_1^{\delta+\vartheta-1}\mathcal{H}(\delta,\vartheta), \end{aligned}$$

When $t_1\rightarrow t_2$, then $|\mathcal{L}(\Psi)(t_2)-\mathcal{L}(\Psi)(t_1)|\rightarrow 0$. As a result, we have

$$\|\mathcal{L}(\Psi)(t_2)-\mathcal{L}(\Psi)(t_1)\|\rightarrow 0,\text{ as }t_1\rightarrow t_2$$

Hence $\mathcal{L}$ is equicontinuous. Then, via the theorem Arzela-Ascoli is continuous. So, via Schauders fixed point result the existence of solutions for the presented system is aproved.

Theorem 2.2. Consider Eq. (11) works and by $\rho<1$, which

$$\rho=\left(\frac{\vartheta\mathbb{T}^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)}+\frac{\delta\vartheta}{\mathcal{N}(\delta)\Gamma(\delta)}\mathbb{T}^{\delta+\vartheta-1}\mathcal{H}(\delta,\vartheta)\right)\mathcal{G}_\Lambda.$$

Then, the existence of solutions for the given model is supported. Proof. For $\Psi,\bar{\Psi}\in\mathcal{B}$, we own

$$
\begin{aligned}
\|\mathcal{L}(\Psi)-\mathcal{L}(\bar{\Psi})\| = & \max_{t\in[0,\mathbb{T}]} \Big| \frac{\vartheta t^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)} \\
& \times (\Lambda(t,\Psi(t))-\Lambda(t,\bar{\Psi}(t))) \\
& + \frac{\gamma\vartheta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^t \omega^{\vartheta-1}(t-\omega)^{\vartheta-1}\,\mathrm{d}\omega \\
& \times [\Lambda(\omega,\Psi(\omega))-\Lambda(\omega,\bar{\Psi}(\omega))] \Big| \\
\leq & \left(\frac{\vartheta\mathbb{T}^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)}+\frac{\delta\vartheta}{\mathcal{N}(\delta)\Gamma(\delta)}\mathbb{T}^{\delta+\vartheta-1}\mathcal{H}(\delta,\vartheta)\right) \\
& \times \|\Psi-\bar{\Psi}\| = \rho\|\Psi-\bar{\Psi}\|,
\end{aligned}
$$

Therefore, $\mathcal{L}$ is a contraction. Then, via the Banach contraction principle, the performed system has solution.

3. Ulam-hyres stability

Now, we prove the Ulam-Hyres stability for the offered system.

Definition 3.1. The offered system is Ulam-Hyres stable if there exists $\aleph_{\delta,\theta} \geq 0$ for $\epsilon > 0$ and for $\Psi \in \mathbb{C}([0,\mathbb{T},\mathbb{R})$ works

$$\left|{}_0^{FFM}D_t^{\delta,\vartheta}\Psi(t)-\Lambda(t,\Psi(t))\right| \leq \epsilon, t\in[0,\mathbb{T}],$$

and there is solution $\Omega \in \mathbb{C}([0,\mathbb{T}],\mathbb{R})$ which

$$|\Psi(t)-\Omega(t)| \leq \aleph_{\delta,\theta}\epsilon, \quad t\in[0,\mathbb{T}]$$

We consider a small disturbance $\Phi \in \mathbb{C}[0,\mathbb{T}]$ which $\Phi(0)=0$. Consider

- $|\Phi(t)| \leq \epsilon$, for $\epsilon > 0$
- ${}_0^{FFM}D_t^{\delta,\theta}\Psi(t) = \Lambda(t,\Psi(t))+\Phi(t)$

Lemma 3.2. The solution for the disturbed system

$$
\begin{aligned}
& {}_0^{FFM}D_t^{\delta,\vartheta}\Psi(t) = \Lambda(t,\Psi(t))+\Phi(t), \\
& \Psi(0) = \Psi_0,
\end{aligned}
$$

accomplish following relation

$$
\begin{aligned}
& \Big| \Psi(t) - \left(\Psi(0)+\frac{\vartheta t^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)}\Lambda(t,\Psi(t))\right. \\
& \left. +\frac{\delta\vartheta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^t \omega^{\vartheta-1}(t-\omega)^{\theta-1}\Lambda(\omega,\Psi(\omega))\mathrm{d}\omega\right) \Big| \leq \Theta_{\delta,\theta}\epsilon
\end{aligned}
\tag{12}
$$

Where $\Theta_{\delta,\vartheta} = \frac{\vartheta\mathbb{T}^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)}+\frac{\delta\vartheta}{\mathcal{N}(\delta)\Gamma(\delta)}\mathbb{T}^{\delta+\vartheta-1}\mathcal{H}(\delta,\vartheta)$.

Proof. The proof is straightforward. One can prove it by assuming Eq. (8). After doing some manipulations on the left-hand side of Eq. (12), the required results would be obtained.

Lemma 3.3. The solutions of considered system is Ulam-Hyres stable, if inequality Eq. (11) holds and $\rho < 1$.

Proof. Suppose we have a unique solution as $\Omega \in \mathcal{B}$ consider $\Psi \in \mathcal{B}$ as the solution of the presented model, therefore

$$
\begin{aligned}
|\Psi(t)-\Omega(t)| = & \Big| \Psi(t) - \left[\Omega(0)+\frac{\vartheta t^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)}\Lambda(t,\Omega(t))\right. \\
& \left.+\frac{\delta\vartheta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^t \omega^{\theta-1}(t-\omega)^{\theta-1}\Lambda(\omega,\Omega(\omega))\mathrm{d}\omega\right] \Big| \\
\leq & \Big| \Psi(t) - \left[\Psi(0)+\frac{\vartheta t^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)}\Lambda(t,\Psi(t))\right. \\
& \left.+\frac{\delta\vartheta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^t \omega^{\vartheta-1}(t-\omega)^{\vartheta-1}\Lambda(\omega,\Psi(\omega))\mathrm{d}\omega\right] \Big| \\
& + \Big| \Psi(0)+\frac{\vartheta t^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)}\Lambda(t,\Psi(t)) \\
& +\frac{\delta\vartheta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^t \omega^{\vartheta-1}(t-\omega)^{\vartheta-1}\Lambda(\omega,\Psi(\omega))\mathrm{d}\omega \Big| \\
& - \Big| \Omega(0)+\frac{\vartheta t^{\vartheta-1}(1-\delta)}{\mathcal{N}(\delta)}\Lambda(t,\Omega(t)) \\
& +\frac{\delta\vartheta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^t \omega^{\vartheta-1}(t-\omega)^{\vartheta-1}\Lambda(\omega,\Omega(\omega))\mathrm{d}\omega \Big| \\
\leq & \Theta_{\delta,\theta}\epsilon + \left(\frac{\vartheta\mathbb{T}^{\theta-1}(1-\delta)}{\mathcal{N}(\delta)}\right. \\
& \left.+\frac{\delta\theta}{\mathcal{N}(\delta)\Gamma(\delta)}\mathbb{T}^{\delta+\theta-1}\mathcal{H}(\delta,\theta)\right) \\
& \times \mathcal{L}_\Lambda|\Psi(t)-\Omega(t)| \\
\leq & \Theta_{\delta,\theta}\epsilon + \rho|\Psi(k)-\Omega(k)|.
\end{aligned}
$$

So, we have $\|\Psi-\Omega\| \leq \Theta_{\delta,\theta}\epsilon+\rho\|\Psi-\Omega\|$

or $\|\Psi-\Omega\| \leq \aleph_{\delta,\theta}\epsilon$

which $\aleph_{\delta,\vartheta} = \frac{\Theta_{\delta,\vartheta}}{1-\rho}$. So, the current system's solution is Ulam-Hyres stable.

4. Numerical results and simulations

Now, let us use the AtanganaToufik method to approximate the solutions of the considered system. This method is an efficient numerical scheme to solve fractional differential equations using initial conditions. To use the Atangana-Toufik method and find an approximate solution of Eq. (2), we firstly rewrite the model as

$$\frac{\mathcal{N}(\delta)}{1-\delta}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S(\omega)E_\delta\left(\frac{-\delta}{1-\delta}(t-\omega)^\delta\right)\mathrm{d}\omega = \vartheta t^{\vartheta-1}\left(B-\vartheta_1 S(t)I(t)-\mu S(t)\right),$$
$$\frac{\mathcal{N}(\delta)}{1-\delta}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t E(\omega)E_\delta\left(\frac{-\delta}{1-\delta}(t-\omega)^\delta\right)\mathrm{d}\omega = \vartheta t^{\vartheta-1}\left(\vartheta_1 S(t)I(t)-(\sigma+\mu+\delta_1)I(t)\right),$$
$$\frac{\mathcal{N}(\delta)}{1-\delta}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t I(\omega)E_\delta\left(\frac{-\delta}{1-\delta}(t-\omega)^\delta\right)\mathrm{d}\omega = \vartheta t^{\vartheta-1}\left(\delta_1 E(t)-(\gamma_1+\mu)I(t)\right),$$
$$\frac{\mathcal{N}(\delta)}{1-\delta}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t R(\omega)E_\delta\left(\frac{-\delta}{1-\delta}(t-\omega)^\delta\right)\mathrm{d}\omega = \vartheta t^{\vartheta-1}\left(\gamma_1 I(t)+\sigma E(t)-\mu R(t)\right)$$

Taking

$$D1(t,S,E,I,R) = \vartheta t^{\vartheta-1}\left(B-\vartheta_1 S(t)I(t)-\mu S(t)\right)$$
$$E1(t,S,E,I,R) = \vartheta t^{\vartheta-1}\left(\vartheta_1 S(t)I(t)-(\sigma+\mu+\delta_1)I(t)\right)$$
$$F1(t,S,E,I,R) = \vartheta t^{\vartheta-1}\left(\delta_1 E(t)-(\gamma_1+\mu)I(t)\right)$$
$$P1(t,S,E,I,R) = \vartheta t^{\vartheta-1}\left(\gamma_1 I(t)+\sigma E(t)-\mu R(t)\right)$$

We have

$$\frac{\mathcal{N}(\delta)}{1-\delta}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S(\omega)E_\delta\left(\frac{-\delta}{1-\delta}(t-\omega)^\delta\right)\mathrm{d}\omega = D1(t,S,E,I,R)$$
$$\frac{\mathcal{N}(\delta)}{1-\delta}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t x(\omega)E_\delta\left(\frac{-\delta}{1-\delta}(t-\omega)^\delta\right)\mathrm{d}\omega = E1(t,S,E,I,R)$$
$$\frac{\mathcal{N}(\delta)}{1-\delta}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t x(\omega)E_\delta\left(\frac{-\delta}{1-\delta}(t-\omega)^\delta\right)\mathrm{d}\omega = F1(t,S,E,I,R)$$
$$\frac{\mathcal{N}(\delta)}{1-\delta}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t x(\omega)E_\delta\left(\frac{-\delta}{1-\delta}(t-\omega)^\delta\right)\mathrm{d}\omega = P1(t,S,E,I,R)$$

Now we use the AB integral on the above system, so we have

$$S(t)-S(0) = \frac{1-\delta}{\mathcal{N}(\delta)}D1(t,S,E,I,R) + \frac{\delta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^t (t-\omega)^{\delta-1}D1(\omega,S,E,I,R)\mathrm{d}\omega,$$
$$E(t)-E(0) = \frac{1-\delta}{\mathcal{N}(\delta)}E1(t,S,E,I,R) + \frac{\delta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^t (t-\omega)^{\delta-1}E1(\omega,S,E,I,R)\mathrm{d}\omega,$$
$$I(t)-I(0) = \frac{1-\delta}{\mathcal{N}(\delta)}F1(t,S,E,I,R) + \frac{\delta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^t (t-\omega)^{\delta-1}F1(\omega,S,E,I,R)\mathrm{d}\omega,$$
$$R(t)-R(0) = \frac{1-\delta}{\mathcal{N}(\delta)}P1(t,S,E,I,R) + \frac{\delta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^t (t-\omega)^{\delta-1}F1(\omega,S,E,I,R)\mathrm{d}\omega.$$

By discretizing at t_{n+1} we get

$$S^{n+1} = S^0 + \frac{1-\delta}{\mathcal{N}(\delta)}D1\left(t_{n+1},S^n,E^n,I^n,R^n\right) + \frac{\delta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^{t_{n+1}}\left(t_{n+1}-\omega\right)^{\delta-1}D1(\omega,S,E,I,R)\mathrm{d}\omega,$$
$$E^{n+1} = E^0 + \frac{1-\delta}{\mathcal{N}(\delta)}E1\left(t_{n+1},S^n,E^n,I^n,R^n\right) + \frac{\delta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^{t_{n+1}}\left(t_{n+1}-\omega\right)^{\delta-1}E1(\omega,S,E,I,R)\mathrm{d}\omega,$$
$$I^{n+1} = I^0 + \frac{1-\delta}{\mathcal{N}(\delta)}F1\left(t_{n+1},S^n,E^n,I^n,R^n\right) + \frac{\delta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^{t_{n+1}}\left(t_{n+1}-\omega\right)^{\delta-1}F1(\omega,S,E,I,R)\mathrm{d}\omega,$$
$$R^{n+1} = R^0 + \frac{1-\delta}{\mathcal{N}(\delta)}P1\left(t_{n+1},S^n,E^n,I^n,R^n\right) + \frac{\delta}{\mathcal{N}(\delta)\Gamma(\delta)}\int_0^{t_{n+1}}\left(t_{n+1}-\omega\right)^{\delta-1}P1(\omega,S,E,I,R)\mathrm{d}\omega.$$

So, results

$$\begin{aligned}S^{n+1} &= S^0 + \frac{1-\delta}{\mathcal{N}(\delta)}D1\left(t_{n+1},S^n,E^n,I^n,R^n\right)\\ &+ \frac{\delta}{\mathcal{N}(\delta)\Gamma(\delta)}\sum_{s=0}^{n}\left[\frac{h^\delta D1\left(t_s,S^n,E^n,I^n,R^n\right)}{\Gamma(\delta+2)}\right.\\ &\times\left.\left((n+1-s)^\delta(n-s+2+\delta)-(n-s)^\delta(n-s+2+\delta)\right)\right]\\ &- \frac{\delta}{\mathcal{N}(\delta)}\sum_{s=0}^{n}\left[\frac{h^\delta D1\left(t_{s-1},S^{n-1},E^{n-1},I^{n-1},R^{n-1}\right)}{\Gamma(\delta+2)}\right.\\ &\times\left.\left((n+1-s)^{\delta+1}-(n-s)^\delta(n-s+1+\delta)\right)\right],\end{aligned}$$

$$\begin{aligned}E^{n+1} &= E^0 + \frac{1-\delta}{\mathcal{N}(\delta)}E1\left(t_{n+1},S^n,E^n,I^n,R^n\right)\\ &+ \frac{\delta}{\mathcal{N}(\delta)\Gamma(\delta)}\sum_{s=0}^{n}\left[\frac{h^\delta E1\left(t_s,S^n,E^n,I^n,R^n\right)}{\Gamma(\delta+2)}\right.\\ &\left.\left((n+1-s)^\delta(n-s+2+\delta)-(n-s)^\delta(n-s+2+\delta)\right)\right]\\ &- \frac{\delta}{\mathcal{N}(\delta)}\sum_{s=0}^{n}\left[\frac{h^\delta E1\left(t_{s-1},S^{n-1},E^{n-1},I^{n-1},R^{n-1}\right)}{\Gamma(\delta+2)}\right.\\ &\left.\left((n+1-s)^{\delta+1}-(n-s)^\delta(n-s+1+\delta)\right)\right]\end{aligned}$$

$$\begin{aligned}I^{n+1} &= I^0 + \frac{1-\delta}{\mathcal{N}(\delta)}F1\left(t_{n+1},S^n,E^n,I^n,R^n\right)\\ &+ \frac{\delta}{\mathcal{N}(\delta)\Gamma(\delta)}\sum_{s=0}^{n}\left[\frac{h^\delta F1\left(t_s,S^n,E^n,I^n,R^n\right)}{\Gamma(\delta+2)}\right.\\ &\left.\left((n+1-s)^\delta(n-s+2+\delta)-(n-s)^\delta(n-s+2+\delta)\right)\right]\\ &- \frac{\delta}{\mathcal{N}(\delta)}\sum_{s=0}^{n}\left[\frac{h^\delta F1\left(t_{s-1},S^{n-1},E^{n-1},I^{n-1},R^{n-1}\right)}{\Gamma(\delta+2)}\right.\\ &\left.\left((n+1-s)^{\delta+1}-(n-s)^\delta(n-s+1+\delta)\right)\right],\end{aligned}$$

$$R^{n+1} = R^0 + \frac{1-\delta}{\mathcal{N}(\delta)} P1\left(t_{n+1}, S^n, E^n, I^n, R^n\right)$$
$$+ \frac{\delta}{\mathcal{N}(\delta)\Gamma(\delta)} \sum_{s=0}^{n} \left[\frac{h^{\delta} P1\left(t_s, S^n, E^n, I^n, R^n\right)}{\Gamma(\delta+2)} \right.$$
$$\left. \times \left((n+1-s)^{\delta}(n-s+2+\delta) - (n-s)^{\delta}(n-s+2+\delta) \right) \right]$$
$$- \frac{\delta}{\mathcal{N}(\delta)} \sum_{s=0}^{n} \left[\frac{h^{\delta} P1\left(t_{s-1}, S^{n-1}, E^{n-1}, I^{n-1}, R^{n-1}\right)}{\Gamma(\delta+2)} \right.$$
$$\left. \times \left((n+1-s)^{\delta+1} - (n-s)^{\delta}(n-s+1+\delta) \right) \right]$$

We show by the gained results undertaking parameters as B = 0.32, $\gamma_1 = 0.2$, $\vartheta_1 = 0.01$, $\mu = 0.2$, $\delta_1 = 0.01$ and $\sigma = 0.25$. Also, conditions S(0) = 1, E(0) = 700, I(0) = 10 and R(0) = 100 are considered. Figs. 1, 2 and 3 are responsible to show 2D simulations and chaotic behavior of solutions using $\delta = 0.95$ and various ϑ. Also, we show the threedimentional solutions for $\delta = 0.95$ through Figs. 4 to 6 for different fractal orders ϑ. Similar behaviours are demonstrated in Figs. 7 to 12 and Figs. 13 to 18 for δ=1 with various ϑ. Effect of fractal order and fractional order on the SEIR model can be brilliantly seen from the reported plots. Moreover, behaviour of the solutions with parameter changes is visible from the reported figures.

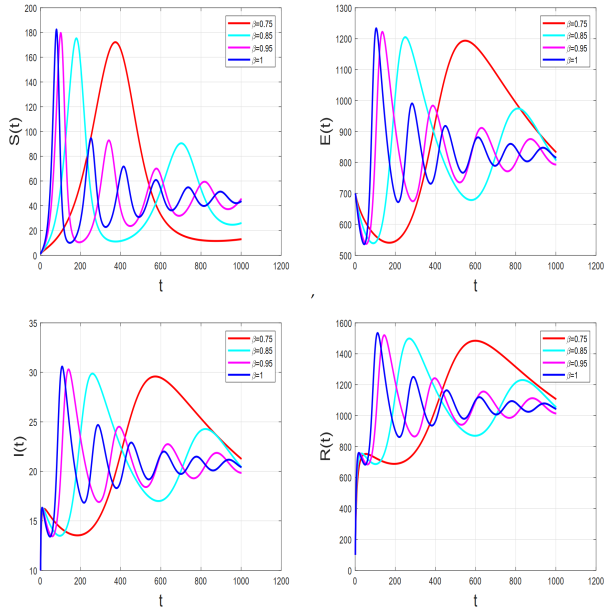

Fig. 1. 2D plot of solutions for δ = 0.95.

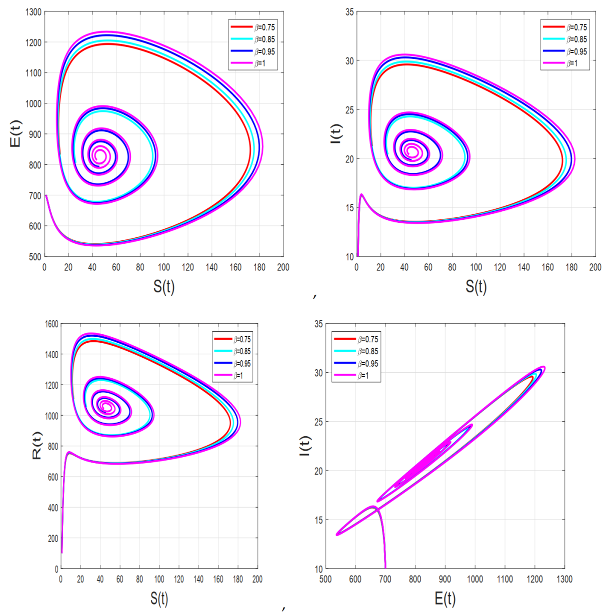

Fig. 2. 2D plot of solutions for δ = 0.95.

5. Conclusion

During this work, numerical simulation of the SEIR system is investigated using fractal fractional derivative in the frame Atangana-Baleanu operator. Also, the theorems of Schauders 65 and Banach's fixed point are used to show the existence of solutions. We discovered the solutions of the Ulam-Hyres stability for the system using non-linear functional analysis. Indeed, Atangana-Toufik technique is used to get the numerical solutions. Figures of solutions for different fractional orders are provided, successfully. To the best of the author's knowledge, this study is the first work to consider the SEIR model by the fractal fractional operator. So, the obtained numerical results, existence theorems, and stability analysis are novel.

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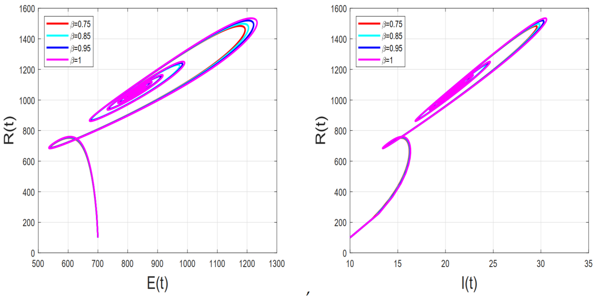

Fig. 3. 2D plot of solutions for $\delta = 0.95$.

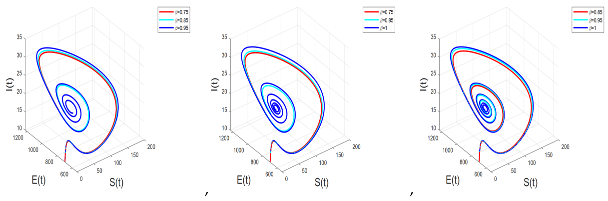

Fig. 4. Chaotic behaviour solutions for $\delta = 0.95$.

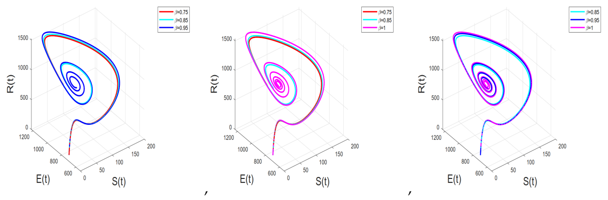

Fig. 5. Chaotic behaviour solutions for $\delta = 0.95$.

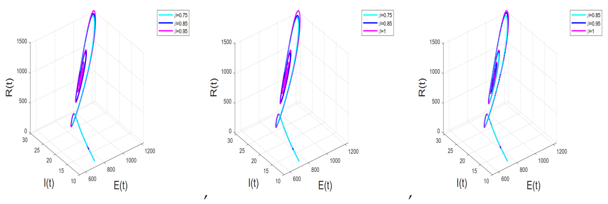

Fig. 6. Chaotic behaviour solutions for $\delta = 0.95$.

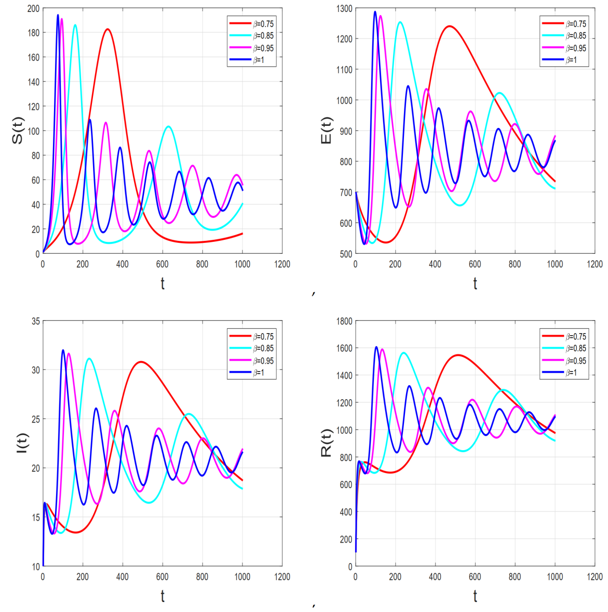

Fig. 7. 2D plot of solutions for $\delta = 0.97$

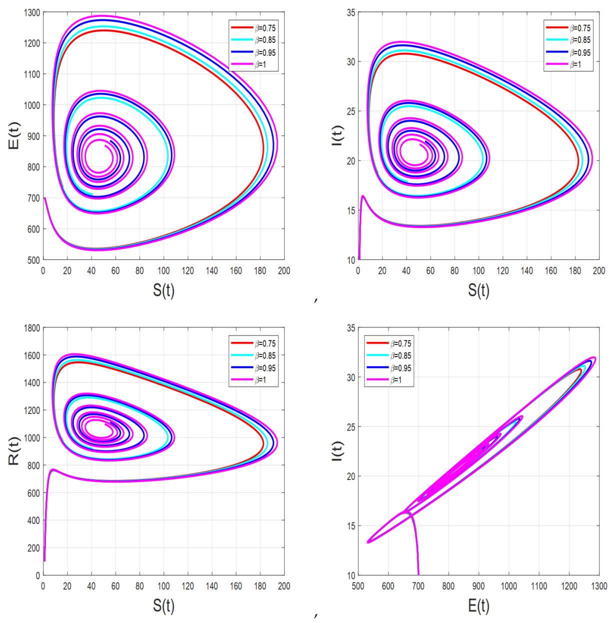

Fig. 8. 2D plot of solutions for $\delta = 0.97$

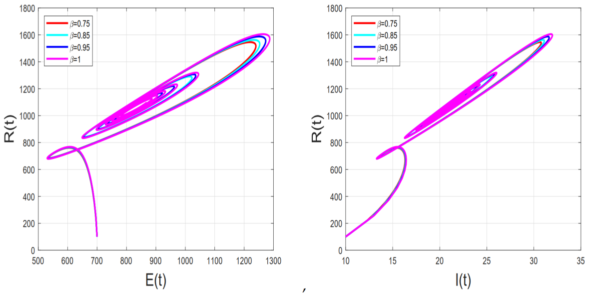

Fig. 9. 2D plot of solutions for $\delta = 0.97$

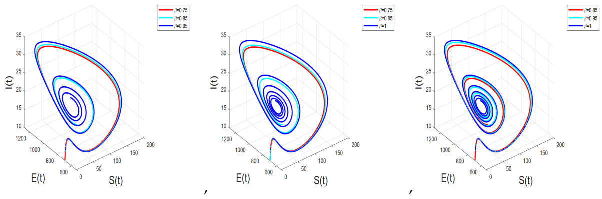

Fig. 10. Chaotic behaviour solutions for $\delta = 0.97$

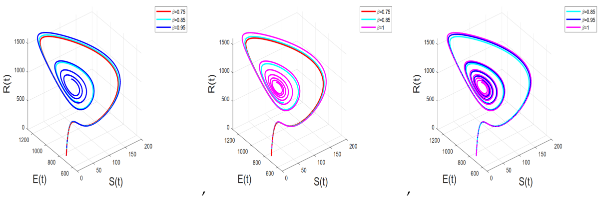

Fig. 11. Chaotic behaviour solutions for $\delta = 0.97$

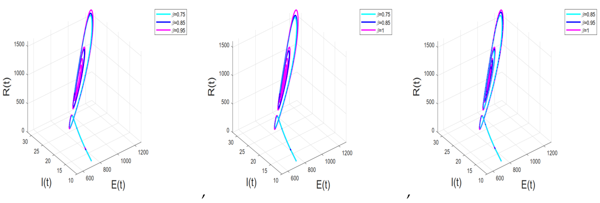

Fig. 12. Chaotic behaviour solutions for $\delta = 0.97$

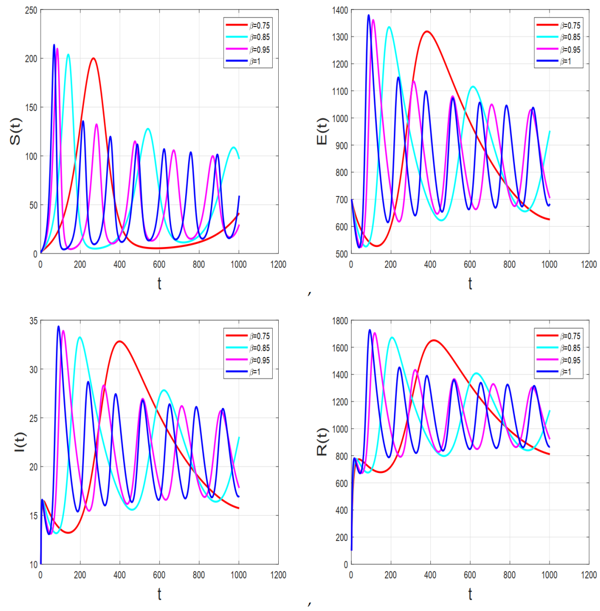

Fig. 13. 2D plot of solutions for $\delta = 1$

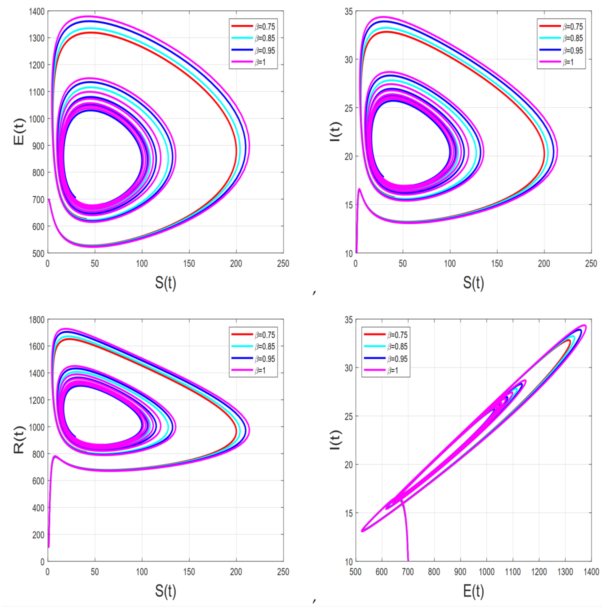

Fig. 14. 2D plot of solutions for $\delta = 1$

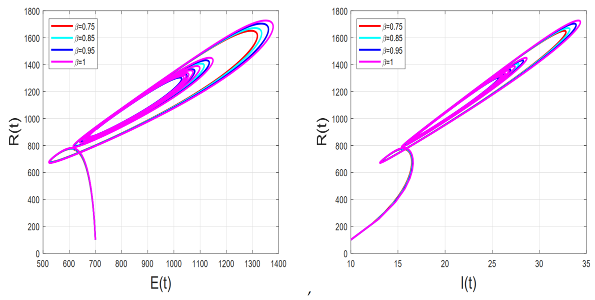

Fig. 15. 2D plot of solutions for $\delta = 1$

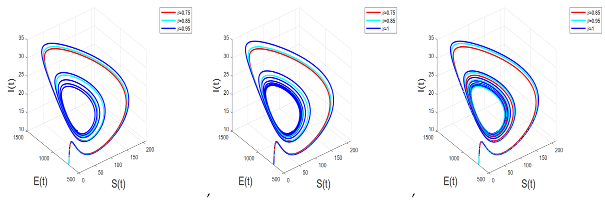

Fig. 16. Chaotic behaviour solutions for $\delta = 1$

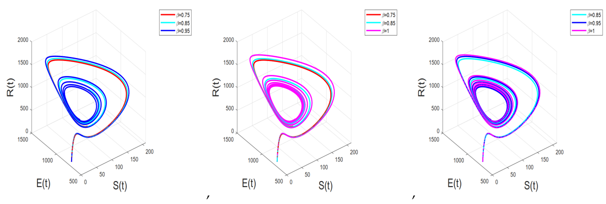

Fig. 17. Chaotic behaviour solutions for $\delta = 1$

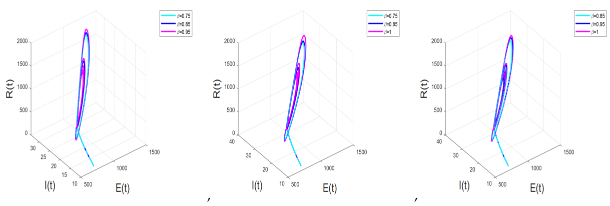

Fig. 18. Chaotic behaviour solutions for $\delta = 1$

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