Parameters and Reliability Estimation for Transmuted Rayleigh Distribution

Jaafer Hmood Eidi[1*] and Makki A. Mohammed Salih[1**]

[1] *Department of Mathematics, College of Education, Almustansiriyah University-Baghdad-Iraq*
[*] *Corresponding author. E-mail: drjaffarmath@uomustansiriyah.edu.iq,*
[**] *Corresponding author. E-mail:dr makki@uomustansiriyah.edu.iq*

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Some estimators were used to estimate parameters and reliability function of the Transmuted Rayleigh (TR) distribution , namely, moments (MOM) and modified moments(MM), least squares (LS), weighted least squares (WLS), new white (NW), and the maximum likelihood Estimation (MLE), where the simulation method was used to generate the required data, where a sample size (n = 10, 50 and 100), repeated sample (N = 1000) and the real value of the parameters for four experiments. The obtained results were compared by mean square error. In general, the results showed that both the of MLE and MOM are the best in estimating scale parameter, but in estimating the transmuted parameter, MLE is the best, and in estimating reliability function, the NW is the better than the rest of the methods.

Keywords: Transmuted Rayleigh distribution, Estimation methods, Simulation, Mean square error.

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1. Introduction

Rayleigh conveyance is generally utilized for endurance investigation. Its application showed up in the clinical sciences (patients endurance times after activity, time of repeat after a medical procedure), designing (part life cycles), computer sciences (disappointment paces of any product framework), showcasing (lifetimes of market rates of parts) and in sociologies (span that graduate per-child stay jobless) [1].The (pdf) and (cdf) of Rayleigh distribution with the location and scale parameters, are given, respectively, as follows [2]:

$$f(x;\mu,\alpha) = 2\alpha(x-\mu)e^{-\alpha(x-\mu)^2} \tag{1}$$

$$F(x,\mu,\alpha) = 1-e^{-\alpha(x-\mu)^2}, x>\mu, \alpha>0 \tag{2}$$

Where (μ) and (α) are location and scale parameters respectively.

Shaw and Buckley proposed as quadratic rank transmutation map [3] :

$$G(x) = F(x)((1+\beta)-\beta F(x)), |\beta| \le 1 \tag{3}$$

$$g(x) = f(x)(1+\beta-2\beta F(x)) \tag{4}$$

Where (β) is the transmuted parameter, $f(x)$ and $F(x)$ are the pdf and of base distribution.

Substitute Eq. (1) and Eq. (2) in Eq. (3) and Eq. (4), we obtained cdf and pdf of TR distribution respectively [1].

$$G(x;\mu,\alpha,\theta) = \left(1-e^{-\alpha(x-\mu)^2}\right)\left(1+\beta e^{-\alpha(x-\mu)^2}\right) \tag{5}$$

$$\begin{aligned} g(x;\mu,\alpha,\theta) = 2\alpha(x-\mu)e^{-\alpha(x-\mu)} \\ (1-\beta+2\beta e^{-\alpha(x-\mu)^2}) \end{aligned} \tag{6}$$

The reliability and hazard functions for TR distribution are:

$$R(x;\mu,\alpha,\theta) = e^{-\alpha(x-\mu)^2}\left(1-\beta\left(1-e^{-\alpha(x-\mu)^2}\right)\right) \tag{7}$$

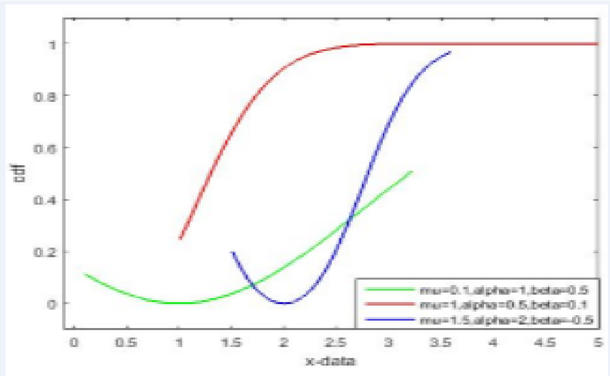

Fig. 1. Plot the cdf distribution.

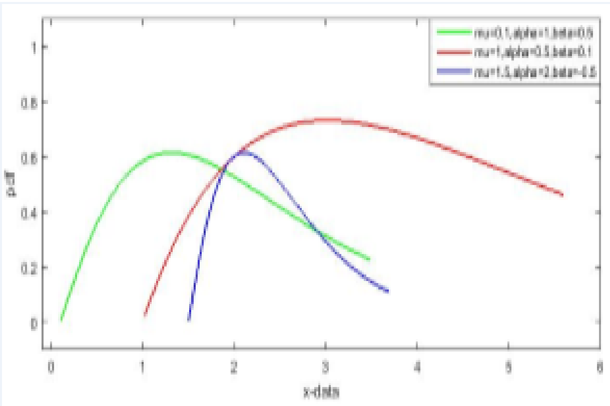

Fig. 2. Plot the pdf distribution.

$$H(x;\mu,\alpha,\theta)=\frac{2\alpha(x-\mu)e^{-\alpha(x-\mu)}\left(1-\beta+2\beta e^{-\alpha(x-\mu)^2}\right)}{e^{-\alpha(x-\mu)^2}\left(1-\beta\left(1-e^{-\alpha(x-\mu)^2}\right)\right)} \tag{8}$$

The Rayleigh distribution is one of the important continuous distributions. As for the TR distribution, it is one of the distributions that have been recently proposed. It is known that the estimation of the parameters of this distribution was not use by the researchers, especially the estimation of the reliability of this distribution. Therefore, we estimated the parameters and reliability of the TR distribution. To achieve the goal of the research, we will derive the estimator formulas for the parameters and the reliability function of this distribution. We will start with how to estimate the initial values for estimating the parameters of this distribution, as follows:

2. Initial value

This method was proposed by Makki and Jaafer (2020) [4] for the importance of the initial values in the estimator formulas for the parameters, as this method depends on finding the initial values on the median value extracted from the generation of data in the simulation and its equality in the equation of the median distribution, as follows:

That the (TR) distribution median is [1]:

$$x_{0.5}=\mu+\left(-\frac{1}{\alpha}\ln\left(\frac{\beta-1\pm\sqrt{\beta^2+1}}{2\beta}\right)\right)^{\frac{1}{2}} \tag{9}$$

From Eq. (9), getting:

$$\alpha_0=-\frac{\ln\left(\frac{\beta-1+\sqrt{\beta^2+1}}{2\beta}\right)}{(x_{0.5}-0.5)^2} \tag{10}$$

And,

$$\beta_0=\frac{\beta-1+\sqrt{\beta^2+1}}{2e^{\left(-\alpha(x_{0.5}-0.5)^2\right)}} \tag{11}$$

The two Eq. (10) and Eq. (11) represent the initial values of the two parameters α and β respectively, and so both α and β are the true values of the parameters and $x_{0.5}$ obtained it from using simulation data.

After completing how to obtain the initial values of the parameters, we will, in the next item, derive the estimations of the parameters through the following estimation methods.

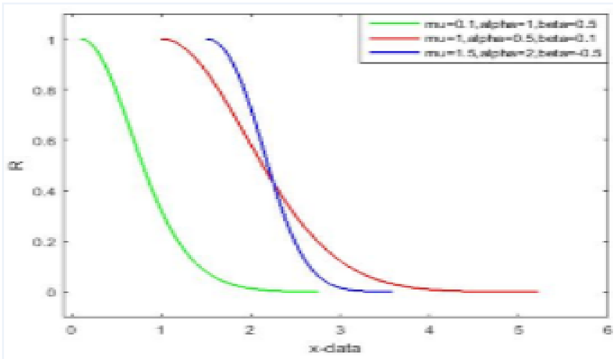

Fig. 3. Plot the Reliability distribution.

3. Estimation methods

3.1. Moments Method

This method depends on the equality the moments of the distribution with the moments of the sample. The first moment is used if the distribution has one parameter, and the second moment is used if the distribution has two parameters and so on... [5].

The first moment for TR distribution is [1]:

$$E(x)=\mu+\frac{\sqrt{\pi}}{2\sqrt{\alpha}}\left(1-\beta+\frac{\beta}{\sqrt{2}}\right) \tag{12}$$

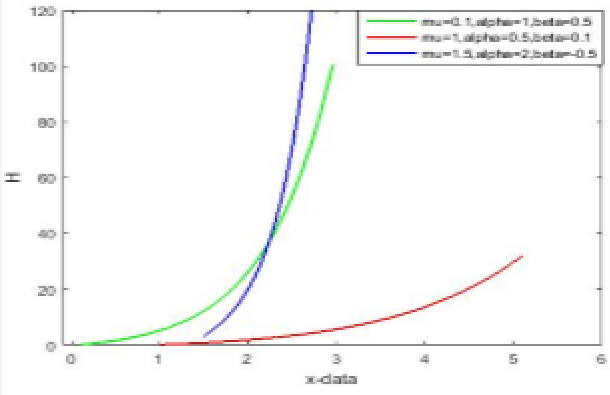

Fig. 4. Plot the Hazard distribution.

From Eq. (12), getting:

$$\left(1-\beta+\frac{\beta}{\sqrt{2}}\right)=\left(\frac{2\sqrt{\alpha}(E(x)-\mu)}{\sqrt{\pi}}\right) \tag{13}$$

Now, numerically and by the fixed point iterative method, from Eq. (13), can be extract many formulas for estimating (β), some formulas may lead to a good estimation, while others may not. For this reason, the following formulas were chosen to estimate the parameter:

$$\beta=1+\frac{\beta}{\sqrt{2}}-\left(\frac{2\sqrt{\alpha}(E(x)-\mu)}{\sqrt{\pi}}\right) \tag{14}$$

Whereas,$(x)=\frac{\sum_{i=1}^{n}x_i}{n}$, then Eq. (14) becomes

$$\hat{\beta}_{MOM}=1+\frac{\beta_0}{\sqrt{2}}-\left(\frac{2\sqrt{\alpha_0}\frac{\sum_{i=1}^{n}x_i}{n}-\mu)}{\sqrt{\pi}}\right) \tag{15}$$

α_0 and β_0 get them from the Eq. (10) and Eq. (11) respectively.

Now, from the second moment, we extract a formula for estimating the parameter α, as follows:

$$E\left(x^2\right)=\mu^2+\frac{\mu\sqrt{\pi}}{\sqrt{\alpha}}\left(1-\beta+\frac{\beta}{\sqrt{2}}\right)+\frac{1}{\alpha}\left(1-\beta+\frac{\beta}{2}\right) \tag{16}$$

From Eq. (15), getting

$$\left(E\left(x^2\right)-\mu^2\right)-\frac{\mu\sqrt{\pi}}{\sqrt{\alpha}}\left(1-\beta+\frac{\beta}{\sqrt{2}}\right)=\frac{1}{\alpha}\left(1-\beta+\frac{\beta}{2}\right) \tag{17}$$

From Eq. (17), the following formula was chosen from several formulas to estimate the parameter α:

$$\alpha=\frac{\left(1-\beta+\frac{\beta}{2}\right)}{\left(E\left(x^2\right)-\mu^2\right)-\frac{\mu\sqrt{\pi}}{\sqrt{\alpha}}\left(1-\beta+\frac{\beta}{\sqrt{2}}\right)} \tag{18}$$

Whereas,$(x^2)=\frac{\sum_{i=1}^{n}x_i^2}{n}$, then Eq. (18) becomes:

$$\widehat{\alpha}_{MOM}=\frac{\left(1-\beta_0+\frac{\beta_0}{2}\right)}{\left(\frac{\sum_{i=1}^{n}x_i^2}{n}-\mu^2\right)-\frac{\mu\sqrt{\pi}}{\sqrt{\alpha_0}}\left(1-\beta_0+\frac{\beta_0}{\sqrt{2}}\right)} \tag{19}$$

α_0 and β_0 can be obtained them from the Eq. (10) and Eq. (11) respectively.

By substituting Eq. (15) and Eq. (19) in Eq. (7), the approximate value of the reliability $\widehat{R}_{MOM}$ can be obtained.

3.2. Modified Moments Method

This method is based on estimating of the parameters on the equality of the expected value of the first order sample for distribution function with the first observation for cdf. The second parameter is estimated by the equality of the sample variance with the variance of the distribution [6].

$$E\left(\widehat{G}\left(x_{(1)}\right)\right)=G\left(x_{(1)}\right) \tag{20}$$

As,

$$E\left(\hat{G}\left(x_{(i)}\right)\right)=P_i=\frac{i}{n+i}, i=1,2,\ldots,n \tag{21}$$

From Eq. (21), if $i=1$, getting:

$$E\left(\widehat{G}\left(x_{(i)}\right)\right)=P_1=\frac{1}{n+1} \tag{22}$$

From Eq. (5), the first order observation $x_{(1)}$ as follows:

By substitute Eq. (5) and Eq. (22) in Eq. (20), the following is obtained:

$$\left(1-e^{-\alpha\left(x_{(1)}-\mu\right)^2}\right)\left(1+\beta e^{-\alpha\left(x_{(1)}-\mu\right)^2}\right)=\frac{1}{n+1}$$

$$1+\beta e^{-\alpha\left(x_{(1)}-\mu\right)^2}-e^{-\alpha\left(x_{(1)}-\mu\right)^2}-\beta e^{-2\alpha\left(x_{(1)}-\mu\right)^2}=\frac{1}{n+1} \tag{23}$$

Numerically, the following formula was chosen from several formulas to estimate the parameter β, as follows:

$$\beta e^{-\alpha\left(x_{(1)}-\mu\right)^2}=\frac{-n}{n+1}+e^{-\alpha\left(x_{(1)}-\mu\right)^2}+\beta e^{-2\alpha\left(x_{(1)}-\mu\right)^2}$$

$$\hat{\beta}_{MM}=\frac{\frac{-n}{n+1}+e^{-\alpha_0\left(x_{(1)}-\mu\right)^2}+\beta_0 e^{-2\alpha_0\left(x_{(1)}-\mu\right)^2}}{e^{-\alpha_0\left(x_{(1)}-\mu\right)^2}} \tag{24}$$

Now, $variance=\frac{1}{\alpha}(1-\beta+\frac{\beta}{2})-\frac{\pi}{4\alpha}(1-\beta+\frac{\beta}{2})^2$

By fixed point iterative method, we chosen the following formula for estimating α

$$\frac{\pi}{4\alpha}(1-\beta+\frac{\beta}{2})^2=\frac{1}{\alpha}(1-\beta+\frac{\beta}{2})-variance$$

$$\hat{\alpha}_{MM} = \frac{\pi\left(1-\beta_0+\frac{\beta_0}{2}\right)^2}{\frac{4}{\alpha_0}\left(1-\beta_0+\frac{\beta_0}{2}\right)-4\text{ variance}} \tag{25}$$

Where as, *variance* $= \frac{\sum_{i=1}^{n}\left(x_{(i)}-\frac{\sum_{i=1}^{n}x_{(i)}}{n}\right)^2}{n-1}$, α_0 and β_0 there valus can be substituted from Eq. (10) and Eq. (11) respectively.

By substituting Eq. (24) and Eq. (25) in Eq. (7), the approximate value of the reliability $\hat{R}_{MM}$ can be obtained.

3.3. Least square Method

This method depends on converting the distribution function into a linear [7]. The technique of this method on a TR distribution shown as follows. From Eq. (5), getting:

$$G = 1+\beta e^{-\alpha(x-\mu)^2} - e^{-\alpha(x-\mu)^2} - \beta e^{-2\alpha(x-\mu)^2}$$

$$G+e^{-\alpha(x-\mu)^2}+\beta e^{-2\alpha(x-\mu)^2}-1 = \beta e^{-\alpha\left(x^2-2\mu\right)}e^{-\alpha\mu^2} \tag{26}$$

By taking natural logarithm for Eq. (26):

$$\ln\left(G+e^{-\alpha(x-\mu)^2}+\beta e^{-2\alpha(x-\mu)^2}-1\right) = \ln(\beta)-\alpha\left(x^2-2\mu\right)-\alpha\mu^2$$

$$\ln\left(G+e^{-\alpha(x-\mu)^2}+\beta e^{-2\alpha(x-\mu)^2}-1\right)+\alpha\mu^2 = \ln(\beta)-\alpha\left(x^2-2\mu\right) \tag{27}$$

The linear equation

$$z_i = a+b\varnothing_i+\epsilon \tag{28}$$

By comparing Eq. (27) with Eq. (28) and replacing G by plotting position (Eq. (21)) , getting:

$$\left.\begin{aligned} z_i = \ln\left(p_i+e^{-\alpha\left(x_{(i)}-\mu\right)^2}+\beta e^{-2\alpha\left(x_{(i)}-\mu\right)^2}-1\right)+\alpha\mu^2 \\ a=\ln(\beta)\rightarrow\beta=e^a \\ b=\alpha \\ \varnothing_i = -\left(x^2-2\mu\right)\end{aligned}\right\} \tag{29}$$

Now, from Eq. (28), getting:

$$\in = z-a-b\varnothing \tag{30}$$

$$\gamma = \sum_{i=1}^{n}\in^2 = \sum_{i=1}^{n}\left(z_{(i)}-a-b\varnothing_{(i)}\right)^2 \tag{31}$$

Now, by taking $\frac{\partial\gamma}{\partial a}$ of Eq. (31) and equating it to zero, the following is obtained:

$$a = \frac{\sum_{i=0}^{n}z_{(i)}-b\sum_{i=0}^{n}\varnothing_{(i)}}{n} \tag{32}$$

Now, by taking $\frac{\partial\gamma}{\partial b}$ of Eq. (31) and equating it to zero, getting:

$$a = \frac{\sum_{i=0}^{n}z_{(i)}\varnothing_{(i)}-b\sum_{i=0}^{n}\varnothing_{(i)}^2}{\sum_{i=0}^{n}\varnothing_{(i)}} \tag{33}$$

By equating Eq. (32) and Eq. (33), getting:

$$\hat{b} = \frac{\sum_{i=0}^{n}z_{(i)}\sum_{i=0}^{n}\varnothing_{(i)}-n\sum_{i=0}^{n}z_{(i)}\varnothing_{(i)}}{\left(\sum_{i=0}^{n}\varnothing_{(i)}\right)^2-n\sum_{i=0}^{n}\varnothing_{(i)}{}^2} \tag{34}$$

From Eq. (31), getting:

$$\hat{\alpha}_{LES} = \hat{b} \tag{35}$$

By substituting $(\hat{b})$, Eq. (34), into Eq. (32), getting $\hat{a}$. From Eq. (29), getting:

$$\hat{\beta}_{LES} = e^{\hat{a}} \tag{36}$$

By substituting Eq. (35) and Eq. (36) in Eq. (7) , the approximate value of the reliability $\hat{R}_{LES}$ is obtained.

3.4. Weighted Least Square Method

This method is very similar to the idea of the least squares method and differs from it by multiplying Eq. (30) by $(\frac{1}{z})$, as the following [8]:

$$\frac{\epsilon}{z} = 1-a\frac{1}{z}-b\frac{\varnothing}{z} \tag{37}$$

Let $\vartheta=\frac{1}{z}$ and $\delta=\frac{\varnothing}{z}$, then Eq. (37) becomes:

$$\frac{\in}{z} = 1-a\vartheta-b\delta \tag{38}$$

By squaring Eq. (38) and taking its summation, getting

$$\omega = \sum_{i=1}^{n}\left(\frac{E}{z}\right)^2 = \sum_{i=1}^{n}\left(1-a\vartheta_{(i)}-b\delta_{(i)}\right)^2 \tag{39}$$

Now, by taking $\frac{\partial\omega}{\partial a}$ of Eq. (39) and equated to zero:

$$a = \frac{\sum_{i=0}^{n}\vartheta_{(i)}-b\sum_{i=0}^{n}\vartheta_{(i)}\delta_{(i)}}{\sum_{i=0}^{n}\vartheta_{(i)}{}^2} \tag{40}$$

Now, by taking $\frac{\partial\omega}{\partial b}$ of Eq. (39) and equating it to zero:

$$a = \frac{\sum_{i=0}^{n}\delta_{(i)}-b\sum_{i=0}^{n}\delta_{(i)}^2}{\sum_{i=0}^{n}\vartheta_{(i)}\delta_{(i)}} \tag{41}$$

From equating Eq. (40) and Eq. (41), getting:

$$\hat{b} = \frac{\sum_{i=0}^{n}\delta_{(i)}\sum_{i=0}^{n}\vartheta_{(i)}{}^2-\sum_{i=0}^{n}\vartheta_{(i)}\sum_{i=0}^{n}\vartheta_{(i)}\delta_{(i)}}{\sum_{i=0}^{n}\delta_{(i)}{}^2\sum_{i=0}^{n}\vartheta_{(i)}{}^2-\left(\sum_{i=0}^{n}\vartheta_{(i)}\delta_{(i)}\right)^2} \tag{42}$$

From Eq. (29) and Eq. (42), the following is obtained:

$$\hat{\alpha}_{WLS} = -\hat{b} \tag{43}$$

By substituting Eq. (43) into Eq. (40), getting $\hat{a}$. Now, substituting $\hat{a}$ into Eq. (29), getting:

$$\hat{\beta}_{WLS} = e^{\hat{a}} \tag{44}$$

By substitute Eq. (43) and Eq. (44) inEq. (7), the approximate value of the reliability $\hat{R}_{WLS}$ is obtained.

3.5. New White Method

This method was proposed by Makki (2006), which depends on the hazard function to find estimates of parameters instead of the reliability function used in White's method [9].

From Eq. (8), getting:

$$\frac{He^{-\alpha(x-\mu)^2}\left(1-\beta\left(1-e^{-\alpha(x-\mu)^2}\right)\right)}{2e^{-\alpha(x-\mu)}} = \alpha(x-\mu)\beta\left(\frac{1}{\beta}-1+2e^{-\alpha(x-\mu)^2}\right)$$

$$\frac{He^{-\alpha(x-\mu)^2}\left(1-\beta\left(1-e^{-\alpha(x-\mu)^2}\right)\right)}{2e^{-\alpha(x-\mu)}} = (\alpha\beta x-\alpha\beta\mu)\left(\frac{1}{\beta}-1+2e^{-\alpha(x-\mu)^2}\right)$$

$$\frac{He^{-\alpha(x-\mu)^2}\left(1-\beta\left(1-e^{-\alpha(x-\mu)^2}\right)\right)}{2e^{-\alpha(x-\mu)}\left(\frac{1}{\beta}-1+2e^{-\alpha(x-\mu)^2}\right)} = -\alpha\beta\mu+\alpha\beta x \tag{45}$$

Comparing Eq. (45) with the following linear equation, getting

$$\rho = c + d\sigma \tag{46}$$

$$\rho_i = \frac{H_0 e^{-\alpha_0\left(x_{(i)}-\mu\right)^2}\left(1-\beta_0\left(1-e^{-\alpha_0\left(x_{(i)}-\mu\right)^2}\right)\right)}{2e^{-\alpha_0\left(x_{(i)}-\mu\right)}\left(\frac{1}{\beta_0}-1+2e^{-\alpha_0\left(x_{(i)}-\mu\right)^2}\right)}$$

where H_0 is the hazard value at initial value of parameters.

$$c = -\alpha\beta_0\mu \tag{47}$$

$$d = \beta \tag{48}$$

$$\sigma_i = \alpha_0 x_{(i)} \tag{49}$$

Then, estimating, getting as the following

$$\hat{d} = \frac{\sum_{i=1}^{n}(\sigma_i-\bar{\sigma})(\rho_i-\bar{\rho})}{\sum_{i=1}^{n}(\sigma_i-\bar{\sigma})^2}$$

From Eq. (48)

$$\hat{\beta}_{NM} = \hat{d} \tag{50}$$

From Eq. (46)

$$\hat{c} = -\bar{\rho} - \hat{d}\bar{\sigma} \tag{51}$$

Where, $\bar{\rho} = \frac{\sum_{i=1}^{n}\rho_i}{n}$ and $\bar{\sigma} = \frac{\sum_{i=1}^{n}\sigma_i}{n}$ By substituting Eq. (51) into Eq. (47), getting:

$$\hat{\alpha}_{NW} = -\frac{\hat{c}}{\beta_0\mu} \tag{52}$$

By substituting Eq. (50) and Eq. (52) in Eq. (7), the approximate value of the reliability $\hat{R}_{NW}$ is obtained.

3.6. Maximum Likelihood Estimation Method (MLE)

This method is considered one of the most widely used methods for estimating parameters due to its consistency, convergence and stability properties. It depends on taking the log-likelihood to pdf for distribution and taking the partial derivation of this function for each parameter of the distribution as follows [10].

The likelihood of the Eq. (6) is

$$L(x_1, x_2, \ldots, x_n; \mu, \alpha, \beta) = 2^n\alpha^n e^{-\alpha\sum_{i=1}^{n}(x_i-\mu)}\prod_{i=1}^{n}(x_i-\mu)\prod_{i=1}^{n}\left(1-\beta+2\beta e^{-\alpha(x_i-\mu)^2}\right) \tag{53}$$

By taking the natural logarithm for Eq. (53), getting

$$\ln L = n\ln(2) + n\ln(\alpha) - \alpha\sum_{i=1}^{n}(x_i-\mu) + \sum_{i=1}^{n}\ln(x_i-\mu) + \sum_{i=1}^{n}\ln(1-\beta+2\beta e^{-\alpha(x_i-\mu)^2}) \tag{54}$$

By taking the partial derivative forEq. (54) w.r.t α and equated to zero, so:

$$\frac{n}{\alpha} = \sum_{i=1}^{n}(x_i-\mu) + \sum_{i=1}^{n}\frac{2\beta(x_i-\mu)^2 e^{-\alpha(x_i-\mu)^2}}{1-\beta+2\beta e^{-\alpha(x_i-\mu)^2}}$$

$$\hat{\alpha}_{MLE} = \frac{n}{\sum_{i=1}^{n}(x_i-\mu) + \sum_{i=1}^{n}\frac{2\beta(x_i-\mu)^2 e^{-\alpha(x_i-\mu)^2}}{1-\beta+2\beta e^{-\alpha(x_i-\mu)^2}}} \tag{55}$$

By taking the partial derivative for Eq. (54) w.r.t β and equated to zero, so:

Table 1. Defult value for parameters

Experiments		E_1	E_2	E_3	E_4
parameters	α	0.5	1	2	1.5
	β	0.1	0.5	1	-.05

$$\frac{\partial \ln L}{\partial \beta} = \sum_{i=1}^{n} \frac{-1+2e^{-\alpha(x_i-\mu)^2}}{1-\beta+2\beta e^{-\alpha(x_i-\mu)^2}} = 0$$

$$\sum_{i=1}^{n} \frac{2e^{-\alpha(x_i-\mu)^2}}{1-\beta+2\beta e^{-\alpha(x_i-\mu)^2}} - \sum_{i=1}^{n} \frac{1}{1-\beta+2\beta e^{-\alpha(x_i-\mu)^2}} = 0$$

$$\frac{1}{\beta}\sum_{i=1}^{\infty} \frac{1}{\frac{1}{\beta}-1+2e^{-\alpha(\alpha_i-\mu)^2}} = \sum_{i=1}^{\infty} \frac{2e^{-\alpha(x_i-\mu)^2}}{1-\beta+2\beta e^{-\alpha(x_i-\mu)^2}}$$

$$\hat{\beta}_{MLE} = \frac{\sum_{i=1}^{n} \frac{1}{\frac{1}{\beta}-1+2e^{-\alpha(x_i-\mu)^2}}}{\sum_{i=1}^{n} \frac{2e^{-\alpha(x_i-\mu)^2}}{1-\beta+2\beta e^{-\alpha(x_i-\mu)^2}}} \quad (56)$$

By substituting and Eq. (56) inEq. (7), the value of the reliability $\hat{R}_{MLE}$ is obtained.

After completing the mathematical derivations of the parameters through the estimation methods in the previous section, we will conduct the experiment aspect of the research by a simulation experiment to generate the data required to estimate these parameters. The next section explains the simulation experience.

4. Simulation Experiment

The simulation experiment depends on three parameters of the distribution, and one of these distribution will be known, which is $\mu = 0.5$, the location parameter was taken as a known value, because this parameter has no direct effect on the shape of the distribution, except that it pulls the shape of the distribution right or left, and many researchers take its value as zero. What is important in this research is to find the estimates of the transformation parameter and what is its effect on the distribution and what are the cases and values that can be taken ,and the other two parameters will be unknown and will be estimated with reliability, namely α and β , through the following stages:

1. The real value of two parameters is shown in the Table 1.

2. Several sizes of samples were taken (n=10, 50 and 100) , and replicate sample (N=1000).

3. Chosen life time for estimating reliability, $x_{(1)} \leq t_{(i)} \leq x_{(n)}$,such that $t_{(i)} = x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ where $x_{(i)}$ random data are generated through the inverse of the cdf for TR distribution from the Eq. (5).

$$x_G = \mu + \left(-\frac{1}{\alpha}\ln\left(\frac{\beta-1\pm\sqrt{\beta^2+1+2\beta-4\beta G}}{2\beta}\right)\right)^{\frac{1}{2}} \quad (57)$$

Assuming U=G, where U represents a regular, continuous random variable known over the period (0,1) , then Eq. (57) becomes as follows: $x_U = \mu + \left(-\frac{1}{\alpha}\ln\left(\frac{\beta-1\pm\sqrt{\beta^2+1+2\beta-4\beta U}}{2\beta}\right)\right)^{\frac{1}{2}}$, using MATLAB language version R2015a.

4. After finding the estimates, they are compared with each other using the comparison mean square error (MSE) criterion as follows: $MSE(\xi) = \frac{\sum_{i=1}^{N}(\hat{\xi}_i-\xi)^2}{N}$, where $(\hat{\xi})$ is any estimator for parameter (ξ).

After completing the stages of the simulation experiment, we will present in the next subsection the results obtained by applying the formulas for estimators of parameters and the reliability function.

4.1. Simulation results

The simulation results are presented of estimation methods (MOM, MM, LS, WLS, NW, and MLE) and their analysis in order to reach the best obtained estimates for parameters and $R(t)$ of TR distribution through comparison between the estimation values depending on the comparison criterion used.

5. Conclusion

- Through the results of the Table 2, it is clear that MOM is the best in estimating α, but in estimating β, it is found that MLE is the best, and in estimating the reliability, it is found that MM is the best compared to the rest of the methods.

- The results of the Table 3 show that MLE is the best in estimating the two parameters α and β, but in estimating reliability, the MLE, MOM and NW shared the advantage.

- The results of the Table 4 show that the best method for estimating α is the MLE, but for estimating β, MM is the best and NW is the best in estimating the reliability function.

Table 2. Estimated values for (α, β) and (R) using (E_1)

Estimators	n	MSE						Best
		Methods						
		MOM	MM	LS	WLS	NW	MLE	
$\hat{\alpha}$	10	0.05466	0.95744	0.13665	0.15494	0.10774	0.09993	MOM
$\hat{\beta}$		0.12040	0.00532	0.01001	0.01795	0.00181	0.00145	MLE
$\hat{R}$		0.00410	0.00072	0.11316	0.15922	0.00041	0.01391	MM
$\hat{\alpha}$	50	0.00747	0.02463	0.13386	0.15296	0.01142	0.07300	MOM
$\hat{\beta}$		0.08715	0.00054	0.00141	0.00301	0.00022	0.00016	MLE
$\hat{R}$		0.00292	0.00010	0.11237	0.12688	0.00002	0.01294	MM
$\hat{\alpha}$	100	0.00312	0.00995	0.12377	0.14189	0.00544	0.07150	MOM
$\hat{\beta}$		0.08543	0.00021	0.00068	0.00166	0.00008	0.00006	MLE
$\hat{R}$		0.00255	0.00007	0.09188	0.10786	0.000008	0.01133	NW

Table 3. Estimated values for (α, β) and (R) using (E_2)

Estimators	n	MSE						Best
		Methods						
		MOM	MM	LS	WLS	NW	MLE	
$\hat{\alpha}$	10	0.62230	5.52968	0.24848	0.53320	0.41244	0.09305	MLE
$\hat{\beta}$		0.17158	0.02976	0.04706	0.06004	0.01951	0.01955	NW
$\hat{R}$		0.00004	0.00482	0.02063	0.08570	0.00024	0.00001	MLE
$\hat{\alpha}$	50	0.04627	0.10944	0.19497	0.48795	0.05278	0.01808	MLE
$\hat{\beta}$		0.02875	0.00352	0.00498	0.01785	0.00285	0.00226	MLE
$\hat{R}$		0.000004	0.00408	0.02030	0.07242	0.00002	0.000005	MOM
$\hat{\alpha}$	100	0.02152	0.07250	0.16696	0.46251	0.02309	0.00832	MLE
$\hat{\beta}$		0.01608	0.00150	0.00260	0.01191	0.00135	0.00100	MLE
$\hat{R}$		0.0000013	0.00406	0.01535	0.06144	0.0000010	0.000003	NW

Table 4. Estimated values for (α, β) and (R) using (E_3)

Estimators	n	MSE						Best
		Methods						
		MOM	MM	LS	WLS	NW	MLE	
$\hat{\alpha}$	10	2.96491	2.54383	1.15065	1.92480	3.64946	0.40604	MLE
$\hat{\beta}$		0.17657	0.03333	0.04866	0.13236	0.65033	0.07617	MM
$\hat{R}$		0.00439	0.02965	0.04335	0.07698	0.00130	0.00792	NW
$\hat{\alpha}$	50	0.25954	0.99777	0.91481	1.91373	1.21505	0.32906	MOM
$\hat{\beta}$		0.12739	0.00552	0.01616	0.09756	0.12447	0.05499	MM
$\hat{R}$		0.00325	0.03162	0.03200	0.06881	0.00025	0.00824	NW
$\hat{\alpha}$	100	0.12515	0.97296	0.90768	1.91141	0.32548	0.31455	MOM
$\hat{\beta}$		0.11273	0.00292	0.00892	0.09681	0.04007	0.02481	MM
$\hat{R}$		0.00280	0.03145	0.02973	0.06639	0.00008	0.00768	NW

Table 5. Estimated values for (α, β) and (R) using (E_4)

Estimators	n	MSE				Best
		Methods				
		MOM	MM	NW	MLE	
$\hat{\alpha}$	10	0.25935	2.24121	0.41714	2.75018	MOM
$\hat{\beta}$		0.25343	0.04999	0.08203	0.02296	MLE
$\hat{R}$		0.00203	0.00340	0.00075	0.01489	NW
$\hat{\alpha}$	50	0.03822	0.26793	0.07500	0.16372	MOM
$\hat{\beta}$		0.09198	0.00748	0.00883	0.00457	MLE
$\hat{R}$		0.00183	0.00329	0.00002	0.00878	NW
$\hat{\alpha}$	100	0.01814	0.14953	0.03152	0.11381	MOM
$\hat{\beta}$		0.07970	0.00350	0.00323	0.00218	MLE
$\hat{R}$		0.00177	0.00270	0.000006	0.00777	NW

- From the Table 5, we note the following: The absence of the two methods of least squares and weighted least squares and the reason is due to the value of the negative transformation parameter. When replacing it with the formulas of the estimators of these two methods, which contain the logarithmic function, the estimations of these methods cannot be found because they will be complex values.

The results of the estimators show that each of MOM, MLE and NW are the best in estimating each of the parameters α, β and the reliability function, respectively.

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