Efficient Caputo-based Solutions For Linear Time-Fractional Partial Differential Equations In Two And Three Dimensions

Iman I. Gorial[1,2*] and Luma N. M. Tawfiq[1]

[1] *Department of. Mathematics /College of Education for Pure Science (Ibn Al-Haitham)/ University of Baghdad*
[2] *Department of General Material Engineering/College of Materials Engineering/ University of Technology*
[*] *Corresponding author. E-mail: eman.esho2403@ihcoedu.uobaghdad.edu.iq; Iman.I.Gorial@uotechnology.edu.iq*

Received: Jan. 03, 2026; Accepted: Apr. 26, 2026

This study presents a new and highly efficient method for solving linear time-fractional partial differential equations (TFPDEs) in two and three dimensions. The proposed method utilizes Caputo fractional derivative to represent the time derivatives. The efficiency of this method is shown by four examples: two-dimensional fractional parabolic equation, three-dimensional fractional diffusion equation, two-dimensional fractional equations, and a three-dimensional fractional heat equation. The outcomes of this study indicate that the proposed method offers accurate and efficient analytical solutions with very simple implementation steps. Therefore, it is suitable to solve a great variety of models in engineering, physics and other scientific fields involving fractional partial differential equations.

Keywords: Riemann-Liouville fractional integral operator, Caputo derivatives, MittageLeffler function, Fractional parabolic equation, Fractional diffusion equation, Fractional heat equation.

http://dx.doi.org/10.6180/jase.202609_32.055

1. Introduction

Fractional differential equations that include fractional derivatives are extensions of classical differential equations that have whole number orders. Fractional partial differential equations (FPDEs) are extensively utilized as models to represent numerous significant physical phenomena, including fluid mechanics, plasma physics, optical fibers, biology, solidstate physics, chemical kinematics, and chemical physics. There is a lot of information concerning fractional differential equations in literature. Because of the difficulty in understanding algebra, there are not many analytical solutions, and just for elementary FDEs. The widely studied subject of fractional calculus has become very important and popular over the past thirty years since it has been shown to be useful in many different areas of science and engineering [1–13]. Before the 19th century, there was no methodology for analytical resolution of fractional differential equations. In the early 20th century, academics focused on the development of long-lasting and stable analysis techniques for exact (or approximate) solution of fractional differential equations [14]. After that, some methods were developed for finding the analytical solutions for the fractional differential equations. These include Adomian decomposition method [15–17], differential transformation method [18–20], symmetry perturbation method [21–24], local fractional variation iteration method [25], finite difference method [26, 27] and the method based on shifted Chebyshev polynomials [28]. To get close to the right answers, most of these methods occasionally need complicated and large-scale calculations. To address these challenges and constraints, the SM method has been presented as a crucial option among all methodologies due to its minimal processing requirements and ease of implementation relative to alternative techniques. The primary motivation for this paper is to propose the SM method for finding analytical solutions to the Linear Time Fractional 2D and 3D Partial Differential Equations. Thing this method, accurate solutions in the form of converging power series can be found rapidly. SM is a powerful, reliable, and highly efficient computational technique for solving applied problems.

This work is structured as follows: it begins with a presentation of the provides essential definitions in fractional calculus necessary for presenting our results, followed by a description of The Suggested Method. This is followed by an analysis of the results and, finally, conclusions are drawn.

2. Method

2.1. Fundamental Concepts

This section introduces the basic definitions and key properties of the Riemann-Liouville fractional integral operator (RLFIO) and Caputo fractional differential operator (CFDO). These preliminary concepts are the theoretical foundation on which the following parts of will be built.

Definition 2.1.[29] Let $f \in C_\alpha$ and $\alpha \geq -1$, then RLFIO of $u(\sigma, t)$ with respect to tof order α is indicted by $J_t^\alpha u(\sigma, t)$ and is explained as:

$$J_t^\alpha u(\sigma, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{(\alpha-1)} u(\sigma, \tau) d\tau, \quad t > 0, \alpha > 0$$

Definition 2.2.[29] Let $n-1 < \alpha \leq n, t \in R$ and $t > 0$. The CFDO for the function $f \in H^1 \Big([a,b], R_{+)}$ with order $\alpha \geq 0$ is explained as:

$$D_t^\alpha u(\sigma, t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u}{\partial \tau^n} d\tau \\ \frac{\partial^n u}{\partial \tau^n}, \quad \alpha = n \in N \end{cases}$$

We have these properties for RLFIO and CFDO:

$$D_t^\alpha t^P = \frac{\Gamma(P+1)}{\Gamma(P-\alpha+1)} t^{(P-\alpha)}$$
$$J_t^\alpha t^P = \frac{\Gamma(P+1)}{\Gamma(P+\alpha+1)} t^{(P+\alpha)}, \quad \alpha > 0, \quad P > -1$$

Keep in mind that the relationship between RLFIO and CFDO is as follows:

$$J_t^\alpha D_t^\alpha u(\sigma, t) = u(\sigma, t) - \sum_{k=0}^{m-1} u^{(k)}(\sigma, t) \frac{t^k}{k!}, \quad n-1 < \alpha < n$$

Definition 2.3.[17] The Mittage-Leffler function for one parameter and two parameters is defined as follows:

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n+1)}, \quad (\alpha \in C, \mathrm{Re}(\alpha) > 0),$$
$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n+\beta)}, \quad (\alpha, \beta \in C, \mathrm{Re}(\alpha, \beta) > 0),$$

When we apply CFDO on MILF we get:

$$D_t^\alpha E_\alpha\left(a t^\alpha\right) = a E_\alpha\left(a t^\alpha\right)$$

Where a is constant.

2.2. Suggested Method Description

In this section, we present Suggested Method to solve linear time fractional partial differential equations. Consider the IVP for linear time FPDE of order $n-1 < \alpha \leq n$

$$L_t^\alpha u(\psi, t) + R(u(\psi, t)) = g(\psi, t) \tag{1}$$

with the initial conditions:

$$\left. \frac{\partial^k u(\psi, t)}{\partial t^k} \right|_{t=0} = f_k(\psi), \quad k = 0, 1, \ldots, n \tag{2}$$

where $L_t^\alpha(u) = \frac{\partial^\alpha}{\partial t^\alpha}, n-1 < \alpha \leq n$, is the Caputo fractional differential operator of highest order fractional derivative with respect to $t, u(\psi, t)$ is unrecognized function which we want to determined, time variable: t, ψ is means variable of space coordinate and sometimes means σ, y in 2D space other means σ, ω, ϱ in 3D space where σ, ω, ϱ are independent space variable, linear differential operator: $R(u)$, nonhomogeneous function: $g(\psi, t)$. In the Suggested Method the unknown dependent function $u(\psi, t)$ can be construed as infinite series of the form:

$$u(\psi, t) = u_0(\psi) + u_1(\psi) t + u_2(\psi) t^2 + \cdots = \sum_{k=0}^{\infty} u_k(\psi) t^k \tag{3}$$

where

$$u_k(\psi) = \frac{1}{k!} \left. \frac{\partial^k u(\psi, t)}{\partial t^k} \right|_{t=0} \tag{4}$$

The next step is to determine the terms $u_n(n = 0, 1, 2, \ldots)$. Now, applying the Riemann-Liouville fractional integral operator of order α on both sides of Eq. (1).

$$J_t^\alpha L_t^\alpha u(\psi, t) + J_t^\alpha R(u(\psi, t)) = J_t^\alpha g(\psi, t) \tag{5}$$

Eq. (5) becomes:

$$u(\psi, t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \left. \frac{\partial^k u(\psi, t)}{\partial t^k} \right|_{t=0} = -J_t^\alpha [R(u)] + J_t^\alpha g(\psi, t) \tag{6}$$

From Eq. (2) obtain that:

$$u(\psi, t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} f_k(\psi) - J_t^\alpha [R(u)] + J_t^\alpha g(\psi, t) \tag{7}$$

where:

$$J_t^{\alpha}(R(u)) = J_t^{\alpha}\left[R\left(\sum_{k=0}^{\infty} u_k(\psi)\, t^k\right)\right]$$
$$= \sum_{k=0}^{\infty} R\left(u_k(\psi)\, \frac{k!}{(\alpha+k)!}\, t^{\alpha+k}\right) \tag{8}$$

Also, the nonhomogeneous term can be written as:

$$G(\psi,t) = J_t^{\alpha}(g(\psi,t)) = \sum_{k=0}^{\infty} g_k \frac{t^k}{k!}$$

Where

$$g_k = \left.\frac{1}{k!}\frac{\partial^k G(\psi,t))}{\partial t^k}\right|_{t=0} \tag{9}$$

Substituting Eq. (8), and Eq. (9) in Eq. Eq. (7) to have:

$$\begin{aligned} u(\psi,t) = & \sum_{k=0}^{n-1} \frac{t^k}{k!} f_k(\psi) \\ & - \sum_{k=0}^{\infty} R\left(u_k(\psi)\frac{k!}{(\alpha+k)!} t^{\alpha+k}\right) \\ & + \sum_{k=0}^{\infty} g_k \frac{t^k}{k!} \end{aligned} \tag{10}$$

$$\because\ j \geq n \quad \text{and} \quad n-1 < \alpha \leq n$$
$$\therefore\ j \geq n \geq \alpha > n-1$$

$$\begin{aligned} u_j(\psi) = & \left.\frac{1}{j!}\frac{\partial^j u(\psi,t)}{\partial t^j}\right|_{t=0} \\ = & \frac{1}{j!}\frac{\partial^j}{\partial t^j}\left[\sum_{k=0}^{n-1}\frac{t^k}{k!} f_k(\psi)\right. \\ & - \sum_{k=0}^{\infty}\frac{k!}{(n+k)!}\Big(R(u_k(\psi))\Big)t^{n+k} \\ & \left. + \sum_{k=0}^{\infty} g_k t^k \right]_{t=0}, \qquad \forall j \geq n \end{aligned}$$

$$\frac{\partial^j}{\partial t^j} t^k = \begin{cases} 0, & k < j \\ \dfrac{k!}{(k-j)!}\, t^{k-j}, & k \geq j \end{cases} \tag{11}$$

Thus Eq. (12) becomes:

$$\begin{aligned} u_j(\psi) = & \frac{1}{j!}\left[-\sum_{k=j-n}^{\infty}\frac{k!}{(n+k)!}\Big(R\,u_j(\psi)\Big)\frac{(n+k)!}{(n+k-j)!}t^{n+k-j}\right. \\ & \left. + \sum_{k=0}^{\infty} g_k \frac{k!}{(k-j)!} t^{k-j}\right]_{t=0} \\ = & \frac{1}{j!}\left[-\frac{(j-n)!}{0!}\Big(R(u_{j-n})\Big) + g_j \frac{j!}{0!} t^{k-j}\right] \end{aligned} \tag{12}$$

Hence

$$u_j(\psi) = g_j - \frac{(j-n)!}{j!}\Big(R\left(u_{j-n}\right)\Big), \quad j \geq n \tag{13}$$

Finally, substituting Eq. (13) in Eq. (3) to get $u(\psi,t)$.

2.3. Convergence

In this subsection, we present the convergence of the proposed method for solving Eq. (1). Theorem 2.1: Let B be a Banach space, then the series $\sum_{k=0}^{\infty} u_k(\psi)t^k$ converges to the solution $u(X,t)$ of Eq. (3) if there exists $0 < Q < 1$ such that:

$$\left\|u_k(\psi)t^k\right\| \leq Q\left\|u_{k-1}(\psi)t^{k-1}\right\|$$

Proof: Define that $\{S_k\}_{k\geq 0}$ is the sequence of partial sums of the series Eq. (3):

$$\{S_k\} = \sum_{n=0}^{k} u_n(\psi)t^n$$

And we need to show that $\{S_k\}_{k\geq 0}$ is a Cauchy sequence in Banach space B. For this purpose, we consider:

$$\begin{aligned} \|S_{k+1} - S_k\| & \leq \left\|u_{k+1}(\psi)t^{k+1}\right\| \\ & \leq Q\left\|u_k(\psi)t^k\right\| \\ & \leq Q^2\left\|u_{k-1}(\psi)t^{k-1}\right\| \\ & \vdots \\ & \leq Q^{k+1}\left\|u_0(\psi)t^0\right\| \end{aligned} \tag{14}$$

Now, for every $k, m \in N, k \geq m$ by using Eq. (14) and triangle inequality successfully, we have

$$\begin{aligned} \|S_k - S_m\| = & \|(S_k - S_{k-1}) + (S_{k-1} - S_{k-2}) \\ & + \cdots + (S_{m+1} - S_m)\| \\ \leq & \|S_k - S_{k-1}\| + \|S_{k-1} - S_{k-2}\| + \cdots \\ & + \|S_{m+1} - S_m\| \\ \leq & Q^k\left\|u_0(\psi)t^0\right\| + Q^{k-1}\left\|u_0(\psi)t^0\right\| + \cdots \\ & + Q^{m+1}\left\|u_0(\psi)t^0\right\| \\ = & Q^{m+1}\left(1 + Q + \cdots + Q^{k-m-1}\right)\left\|u_0(\psi)t^0\right\| \\ \leq & Q^{m+1}\left(\frac{1-Q^{k-m}}{1-Q}\right)\left\|u_0(\psi)t^0\right\| \end{aligned} \tag{15}$$

Since $0 < Q < 1$, it follows that $1 - Q^{k-m} < 1$, and thus we get the following:

$$\|S_k - S_m\| \leq \frac{Q^{m+1}}{1-Q}\left\|u_0(\psi)t^0\right\|$$

Then we have

$$\lim_{k,m\to\infty}\|S_k - S_m\| = 0 \tag{16}$$

Therefore, $\{S_k\}_{k\geq 0}$ is a Cauchy sequence in the Banach space and it implies that the series solution:

$$u(\psi,t) = \sum_{k=0}^{\infty} u_k(\psi)t^k \tag{17}$$

It is convergent. This completes the proof of the theorem.

3. Results and discussion

3.1. Explanatory examples

This section shows several examples to illustrate the application of the proposed method to solving linear time-fractional partial differential equations. These examples are given to demonstrate the procedure for implementation and show the effectiveness of the approach adopted. Example 1. we consider parabolic equation [30–32]:

$D_t^\alpha u(\sigma,\omega,t) + 2\left(\frac{1}{\sigma^2}+\frac{\sigma^4}{6!}\right)\frac{\partial^4 u(\sigma,\omega,t)}{\partial\sigma^4} + 2\left(\frac{1}{\omega^2}+\frac{\omega^4}{6!}\right)\frac{\partial^4 u(\sigma,\omega,t)}{\partial\omega^4} = 0,\quad 1 < \alpha \leq 2$ subject to initial condition: $u(\sigma,\omega,0) = 0,\quad u_t(\sigma,\omega,0) = 2+\frac{\sigma^6}{6!}+\frac{\omega^6}{6!}$

It is clear that: $R(u) = 2\left(\frac{1}{\sigma^2}+\frac{\sigma^4}{6!}\right)u_{\sigma\sigma\sigma\sigma} + 2\left(\frac{1}{\omega^2}+\frac{\omega^4}{6!}\right)u_{\omega\omega\omega\omega}, N(u) = 0, g(\sigma,\omega,t) = 0$ then $g_k = 0, \forall k = 0,1,2,\ldots$

From ICs $u_0 = 0$ and $u_1 = 2+\frac{\sigma^6}{6!}+\frac{\omega^6}{6!}$
From Eq. (13) we get:

$$\begin{aligned} u_2(\sigma,\omega) &= g_2 - \frac{(2-2)!}{2!}\left(R\left(u_{2-2}(\sigma)\right)\right)\\ &= 0 - \frac{1}{2!}2\left(\frac{1}{\sigma^2}+\frac{\sigma^4}{6!}\right)u_{0\sigma\sigma\sigma\sigma}\\ &+2\left(\frac{1}{\omega^2}+\frac{\omega^4}{6!}\right)u_{0\omega\omega\omega\omega} \qquad = 0\end{aligned} \tag{18}$$

$$\begin{aligned} u_3(\sigma,\omega) &= g_3 - \frac{(3-2)!}{3!}\left(R\left(u_{3-2}(\sigma)\right)\right)\\ &= 0 - \frac{1}{3!}2\left(\frac{1}{\sigma^2}+\frac{\sigma^4}{6!}\right)u_{1\sigma\sigma\sigma\sigma}\\ &+2\left(\frac{1}{\omega^2}+\frac{\omega^4}{6!}\right)u_{1\omega\omega\omega\omega}\\ &= \frac{-1}{3!}2\left(\frac{1}{\sigma^2}+\frac{\sigma^4}{6!}\right)(\frac{\sigma^2}{2}\\ &+2\left(\frac{1}{\omega^2}+\frac{\omega^4}{6!}\right)\left(\frac{\omega^2}{2}\right)\\ &= \frac{-1}{3!}\left(2+\frac{\sigma^6}{6!}+\frac{\omega^6}{6!}\right)\end{aligned} \tag{19}$$

$$\begin{aligned} u_4(\sigma,\omega) &= g_4 - \frac{(4-2)!}{4!}\left(R(u_{4-2}(\sigma))\right)\\ &= 0 - \frac{2!}{3!}\left(2\left(\frac{1}{\sigma^2}+\frac{\sigma}{6!}\right)u_{2,\sigma\sigma\sigma\sigma}\right.\\ &\left.+2\left(\frac{1}{\omega^2}+\frac{\omega^4}{6!}\right)u_{2,\omega\omega\omega\omega}\right)\\ &= 0\end{aligned} \tag{20}$$

$$\begin{aligned} u_5(\sigma,\omega) &= g_5 - \frac{(5-2)!}{5!}\left(R(u_{5-2}(\sigma))\right)\\ &= 0 - \frac{3!}{5!}\left(2\left(\frac{1}{\sigma^2}+\frac{\sigma^4}{6!}\right)u_{3,\sigma\sigma\sigma\sigma}\right.\\ &\left.+2\left(\frac{1}{\omega^2}+\frac{\omega^4}{6!}\right)u_{3,\omega\omega\omega\omega}\right)\\ &= \frac{-3!}{5!}\left(\frac{-1}{3!}\left(2\left(\frac{1}{\sigma^2}+\frac{\sigma^4}{6!}\right)\left(\frac{\sigma^2}{2}\right)\right.\right.\\ &\left.\left.+2\left(\frac{1}{\omega^2}+\frac{\omega^4}{6!}\right)\left(\frac{\omega^2}{2}\right)\right)\right)\\ &= \frac{1}{5!}\left(2+\frac{\sigma^6}{6!}+\frac{\omega^6}{6!}\right)\end{aligned} \tag{21}$$

Then by Eq. (3):

$$\begin{aligned} u(\sigma,\omega,t) &= \left(2+\frac{\sigma^6}{6!}+\frac{\omega^6}{6!}\right)t - \frac{t^3}{3!}\left(2+\frac{\sigma^6}{6!}+\frac{\omega^6}{6!}\right)\\ &+\frac{t^5}{5!}\left(2+\frac{\sigma^6}{6!}+\frac{\omega^6}{6!}\right)+\cdots\\ &= \left(2+\frac{\sigma^6}{6!}+\frac{\omega^6}{6!}\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}+\cdots\right)\\ &= \left(2+\frac{\sigma^6}{6!}+\frac{\omega^6}{6!}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{\Gamma(2k+2)}\\ &= \left(2+\frac{\sigma^6}{6!}+\frac{\omega^6}{6!}\right)tE_{2,2}\left(-t^2\right)\end{aligned} \tag{22}$$

So, the solution is close to exact as the form: $u(\sigma,\omega,t) = \left(2+\frac{\sigma\sigma^6}{6!}+\frac{\omega^6}{6!}\right)\sin t$ Table 1, illustrate a comparison between the numerical result obtained from Suggested Method and results obtained by others methd such ADM in [30], HAM in [31], HVIM in [32], and exact solution for different values of t, σ and $\omega = 0.4$, while Fig. 1, illustrated a comparison between exact solution and solution Suggested Method at $\sigma = \omega = 0.3$ with different values of t and $\alpha = 2$. But Fig. 2, illustrated the accuracy of Suggested Method when $\sigma = \omega = 0.3$, and different values of α.

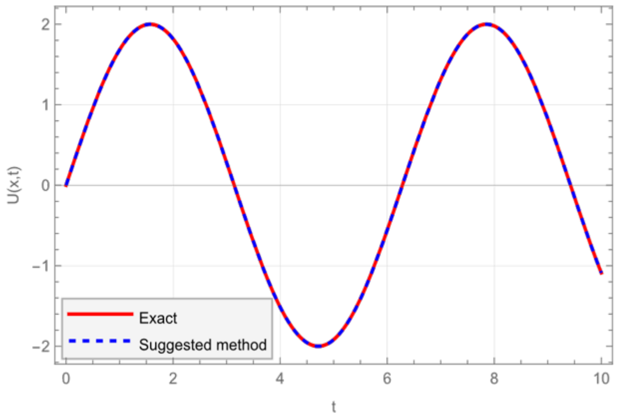

Fig. 1. Suggested Method and Exact Solution for Example 1, $\sigma = \omega = 0.3$

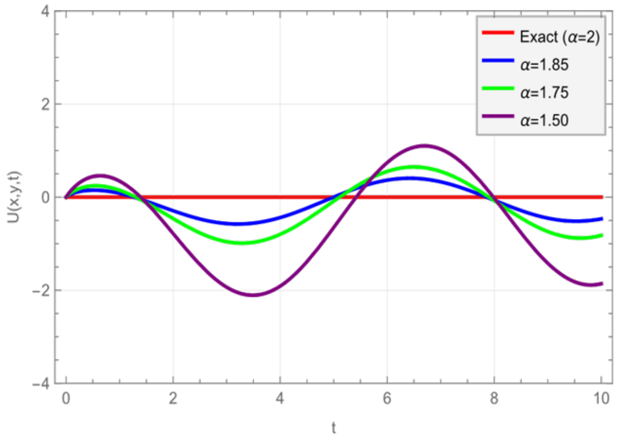

Fig. 2. Accuracy of the results, for example 1, $\sigma = \omega = 0.3$

Example 2. Consider 3D-fractional order diffusion equation [33]:

$$D_t^{\alpha} u(\sigma,\omega,\varrho,t) = u_{\sigma\sigma}(\sigma,\omega,\varrho,t) + u_{\omega\omega}(\sigma,\omega,\varrho,t) + u_{zz}(\sigma,\omega,\varrho,t), \qquad 0 < \alpha \leq 1 \tag{23}$$

subject to initial condition: $u(\sigma,\omega,\varrho,0) = \sin\sigma\sin\omega\sin\varrho$

It is clear that: $R(u) = u_{\sigma\sigma} + u_{\omega\omega} + u_{\varrho\varrho}, N(u) = 0, g(\sigma,\omega,\varrho,t) = 0$ then $g_k = 0, \forall k = 0,1,2,\dots$

From ICs $\quad u_0 = \sin\sigma\sin\omega\sin\varrho$

From Eq. (13) we get:

$$\begin{aligned} u_1(\sigma,\omega,\varrho) &= g_1 + \frac{(1-1)!}{1!}\left(R\left(u_{1-1}\right)\right) \\ &= 0 + \frac{1}{1!}\left(u_{0\sigma\sigma} + u_{0\omega\omega} + u_{0\varrho\varrho}\right) \\ &= \frac{1}{1!}(-\sin\sigma\sin\omega\sin\varrho - \sin\sigma\sin\omega\sin\varrho \\ &\quad - \sin\sigma\sin\omega\sin\varrho) \\ &= -3\sin\sigma\sin\omega\sin\varrho \end{aligned} \tag{24a}$$

$$\begin{aligned} u_2(\sigma,\omega,\varrho) &= g_2 + \frac{(2-1)!}{2!}\left(R\left(u_{2-1}(\sigma)\right)\right) \\ &= 0 + \frac{1}{2!}\left(u_{1\sigma\sigma} + u_{1\omega\omega} + u_{1\varrho\varrho}\right) \\ &= \frac{1}{2!}(3\sin\sigma\sin\omega\sin\varrho + 3\sin\sigma\sin\omega\sin\varrho \\ &\quad +3\sin\sigma\sin\omega\sin\varrho) \\ &= \frac{9}{2!}\sin\sigma\sin\omega\sin\varrho \end{aligned} \tag{24b}$$

$$\begin{aligned} u_3(\sigma,\omega,\varrho) &= g_3 + \frac{(3-1)!}{3!}\left(R\left(u_{3-1}(\sigma)\right)\right) \\ &= 0 + \frac{2!}{3!}\left(u_{2\sigma\sigma} + u_{2\omega\omega} + u_{2\varrho\varrho}\right) \\ &= \frac{2!}{3!}\left(-\frac{9}{2!}\sin\sigma\sin\omega\sin\varrho - \frac{9}{2!}\sin\sigma\sin\omega\sin\varrho \right. \\ &\quad \left. -\frac{9}{2!}\sin\sigma\sin\omega\sin\varrho\right) \\ &= \frac{2!}{3!}\left(-\frac{27}{2!}\sin\sigma\sin\omega\sin\varrho\right) \\ &= -\frac{27}{3!}\sin\sigma\sin\omega\sin\varrho \end{aligned} \tag{24c}$$

Then by Eq. (3), we have:

$$\begin{aligned} u(\sigma,y,\varrho,t) = {} & (\sin\sigma\sin\omega\sin\varrho) \\ & - \frac{3t}{1!}(\sin\sigma\sin\omega\sin\varrho) \\ & + \frac{9t^2}{2!}(\sin\sigma\sin\omega\sin\varrho) \\ & - \frac{27t^3}{3!}(\sin\sigma\sin\omega\sin\varrho) + \cdots \end{aligned} \tag{25}$$

$$= \sin\sigma\sin\omega\sin\varrho\left(1 - \frac{3t}{1!} + \frac{(3t)^2}{2!} - \frac{(3t)^3}{3!} + \cdots\right)$$

$$= \sin\sigma\sin\omega\sin\varrho\sum_{k=0}^{\infty}\frac{(-3t)^k}{\Gamma(2k+1)}$$

$$= \sin\sigma\sin\omega\sin\varrho E_{1,1}(-3t)$$

Table 1. Numerical Simulation Results Derived from the Suggested Method (SM) Technique for Example 1, $\mathrm{u}(\sigma, 0.4, \mathrm{t})$ corresponding to $\alpha = 2$

t	σ	u_{ADM}	u_{HVIM}	u_{HAM}	u_{SM}	U_{Exact}
0.	0.6	0.3973526655714562	0.3973526655803831	0.3973526655705066	0.3973526655705064	0.3973526655705064
	0.7	0.3973722546483623	0.3973722546746382	0.3973722546424527	0.3973722546424525	0.3973722546424525
	0.8	0.3974121250903782	0.3974121250943579	0.3974121250937766	0.3974121250937764	0.3974121250937764
	1.0	0.3976157214292649	0.3976157214257291	0.3976157214239754	0.3976157214239752	0.3976157214239752
0.	0.6	0.7788641342878782	0.7788641342456280	0.7788641342835644	0.7788641342835634	0.7788641342835634
	0.7	0.7789025314778304	0.7789025314143790	0.7789025314729745	0.7789025314729737	0.7789025314729737
	0.8	0.7789806828635426	0.7789806828473879	0.7789806828665466	0.7789806828665459	0.7789806828665459
	1.0	0.7793797587884732	0.7793797587748378	0.7793797587837445	0.7793797587837438	0.7793797587837438
0.	0.6	1.1293247478021349	1,1293247478874638	1.1293247478106380	1.1293247478106399	1.1293247478106399
	0.7	1.1293804223420383	1.1293804223367422	1.1293804223427404	1.1293804223427422	1.1293804223427422
	0.8	1.1294937390248782	1.1294937390008437	1.1294937390291210	1.1294937390291218	1.1294937390291218
	1.0	1.1300723846384273	1.1300723846123892	1.1300723846358554	1.1300723846358571	1.1300723846358571

So, the solution is close to exact as the form: $u(\sigma, \omega, \varrho, t) = e^{-3t} \sin\sigma \sin\omega \sin\varrho$ Is same as obtained by [33]

Table 2 illustrate a comparison between the numerical result obtained from suggested method and exact solution for different values of $\mathrm{t}, \sigma, \omega, \varrho$. while Fig. 3, illustrated a comparison between exact solution and solution Suggested Method at $\sigma = \omega = \varrho = 0.4$ with different values of t and $\alpha = 1$. But Fig. 4, illustrated the accuracy of Suggested Method when $\sigma = \omega = \varrho = 0.4$, and different values of α.

Example 3. we consider the two-dimensional fractional equations [34]:

$$D_t^{\alpha} u(\sigma, \omega, t) = 2\left(\frac{\partial^2 u(\sigma, \omega, t)}{\partial \sigma^2} + \frac{\partial^2 u(\sigma, \omega, t)}{\partial \omega^2}\right), \quad 1 < \alpha \le 2 \tag{26}$$

subject to initial condition: $u(x, \omega, 0) = \sin x \sin\omega, \quad u_t(x, \omega, 0) = 0$

It is clear that: $R(u) = 2\left(u_{\sigma\sigma} + u_{\omega\omega}\right), N(u) = 0, g(\sigma, \omega, t) = 0$ then $g_k = 0, \forall k = 0, 1, 2, \ldots$

From ICs $u_0 = \sin\sigma \sin\omega$ and $u_1 = 0$

From Eq. (13) we get:

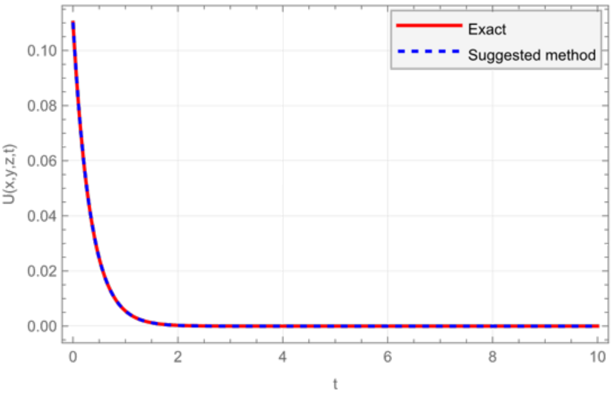

Fig. 3. Suggested Method and Exact Solution for Example 2, when $\sigma = \omega = \varrho = 0.4$

$$\begin{aligned}
u_2(\sigma, \omega) &= g_2 + \frac{(2-2)!}{2!}\left(R(u_{2-2}(\sigma))\right) \\
&= 0 + \frac{1}{2!}\left(2\left(u_{0,\sigma\sigma} + u_{0,\omega\omega}\right)\right) \\
&= \frac{1}{2!}\left(2(-\sin\sigma\sin\omega - \sin\sigma\sin\omega)\right) \\
&= -\frac{1}{2!}(4\sin\sigma\sin\omega) \\
u_3(\sigma, \omega) &= g_3 + \frac{(3-2)!}{3!}\left(R(u_{3-2}(\sigma))\right) \\
&= 0 + \frac{1}{3!}\left(2\left(u_{1,\sigma\sigma} + u_{1,\omega\omega}\right)\right) = 0 \\
u_4(\sigma, \omega) &= g_4 + \frac{(4-2)!}{4!}\left(R(u_{4-2}(\sigma))\right) \\
&= 0 + \frac{1}{2!}\left(2\left(u_{2,\sigma\sigma} + u_{2,\omega\omega}\right)\right) \\
&= \frac{2!}{4!}\left(2\left(\frac{-4}{2!}(-\sin\sigma\sin\omega - \sin\sigma\sin\omega)\right)\right) \\
&= \frac{1}{4!}(16\sin\sigma\sin\omega)
\end{aligned} \tag{27}$$

Table 2. Absolute Error Comparison Between the Computed Numerical Solutions and the Exact Solution for Example 4 at $\sigma = \omega = \varrho = 0.2, 0.4, 0.6$

t	σ	$\alpha = 1$	u_{Exact}	Absolute error $\alpha = 1$
	0.2	0.005809036992260071	0.005809036992260071	0.0000000000000000
	0.4	0.043748268269425120	0.043748268269425120	0.0000000000000000
	0.6	0.133362057065803000	0.133362057065803000	0.0000000000000000
0.3	0.2	0.003188067095852973	0.003188067095852973	0.0000000000000000
	0.4	0.024009558685223586	0.024009558685223586	0.0000000000000000
	0.6	0.073190648731148290	0.073190648731148290	0.0000000000000000
0.5	0.2	0.001749648318852601	0.001749648318852601	0.0000000000000000
	0.4	0.013176725183933096	0.013176725183933096	0.0000000000000000
	0.6	0.040167879676924666	0.040167879676924666	0.0000000000000000

Then by Eq. (3), we have:

$$
\begin{aligned}
u(\sigma,\omega,t) &= (\sin\sigma\sin\omega) - \frac{t^2}{2!}(4\sin\sigma\sin\omega) \\
&\quad + \frac{t^4}{4!}(16\sin\sigma\sin\omega) + \cdots \\
&= \sin\sigma\sin\omega\left(1 - \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} + \cdots\right) \\
&= \sin\sigma\sin\omega\sum_{k=0}^{\infty}\frac{(-4t^2)^k}{\Gamma(2k+1)} \\
&= \sin\sigma\sin\omega\ E_{2,1}\left(-4t^2\right)
\end{aligned}
\tag{28}
$$

The exact solution in closed form is: $u(\sigma,\omega,t) = \sin\sigma\sin\omega\cos 2t$ Is it the same as obtained by [34] Table 3 illustrate a comparison between the numerical result obtained from Suggested Method exact solution evaluated at various values of t, σ, and ω. Fig. 5 presents a comparison between the exact solution and the solution obtained by the proposed method at $\sigma = \omega = 0.2$ for various values of t with $\alpha = 2$. Fig. 6demonstrates the accuracy of the proposed method at $\sigma = \omega = 0.2$ for different values of α

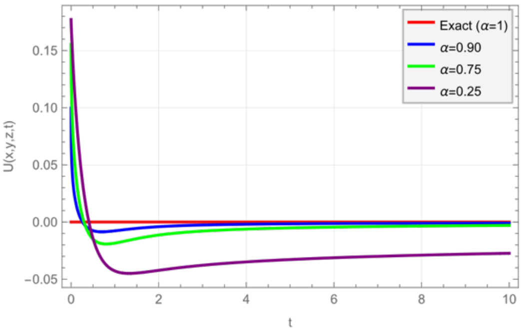

Fig. 4. Accuracy of the results, for example 2, $\sigma = \omega = \varrho = 0.4$

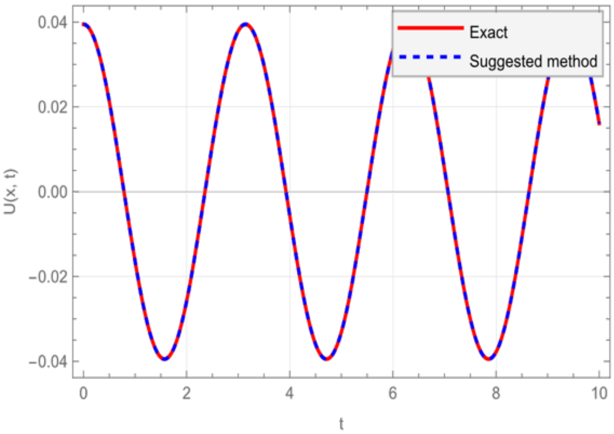

Fig. 5. Suggested Method and Exact Solution for Example 3, $\sigma = \omega = 0.2$

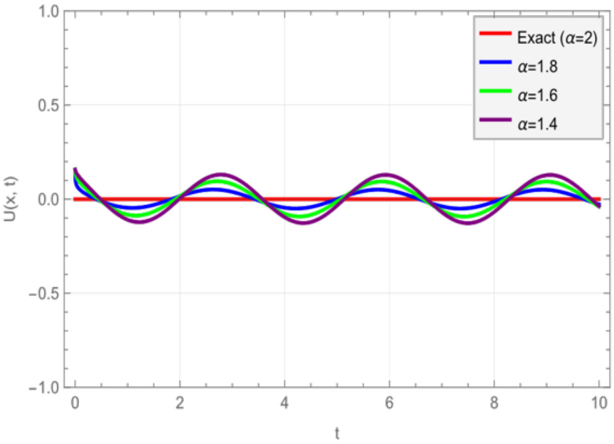

Fig. 6. Accuracy of the results, for example 3, $\sigma = \omega = 0.2$

3.2. Discussions of Stability

We found the closed form solutions of the fractional time partial differential equations in two and three dimensions. The recommended method works out because the computed solution patterns agree with the known precise solutions.

Table 3. Absolute Error Comparison Between the Computed Numerical Solutions and the Exact Solution for Example 3 at $\sigma = \omega = 0.2, 0.3, 0.4$

t	σ	u_{SM}	u_{Exact}	Absolute error $\alpha = 1$
0.2	0.2	0.03635381966465119	0.03635381966465119	0.0000000000000000
	0.3	0.08043827607409718	0.08043827607409718	0.0000000000000000
	0.4	0.13967580988155290	0.13967580988155290	0.0000000000000000
0.4	0.2	0.02749866755369304	0.02749866755369304	0.0000000000000000
	0.3	0.06084492448821206	0.06084492448821206	0.0000000000000000
	0.4	0.10565323524890490	0.10565323524890490	0.0000000000000000
0.6	0.2	0.01430208047686762	0.01430208047686762	0.0000000000000000
	0.3	0.03164549718418898	0.03164549718418898	0.0000000000000000
	0.4	0.05495033787440114	0.05495033787440114	0.0000000000000000

Table 4. Absolute Error Comparison Between the Computed Numerical Solutions and the Exact Solution for Example 4 at $\sigma = \omega = \varrho = 0.25, 0.50, 1.$

σ	t	u_{NTDM} [35]	u_{SM}	u_{Exact}	Absolute Error of SM
0.2	0.5	0.386654×10^{-7}	$-2.345260025783578 \times 10^{-8}$	$-2.345260025783578 \times 10^{-8}$	0.0000000000000000000
	0.7	0.665619×10^{-7}	$-3.144940419548662 \times 10^{-8}$	$-3.144940419548662 \times 10^{-8}$	0.0000000000000000000
	1	0.102321×10^{-6}	$-3.767732136419759 \times 10^{-8}$	$-3.767732136419759 \times 10^{-8}$	0.0000000000000000000
0.5	0.5	0.0001583735	-0.0000960618506560954	-0.0000960618506560954	0.0000000000000000000
	0.7	0.0002726376	-0.0001288167595584713	-0.0001288167595584713	0.0000000000000000000
	1	0.000419108	-0.0001543263083077530	-0.0001543263083077530	0.0000000000000000000
1	0.5	0.6486979167	-0.3934693402873666000	-0.3934693402873666000	0.0000000000000000000
	0.7	1.1167236330	-0.5276334472589853000	-0.5276334472589853000	0.0000000000000000000
	1	1.7166666670	-0.6321205588285577000	-0.6321205588285577000	0.0000000000000000000

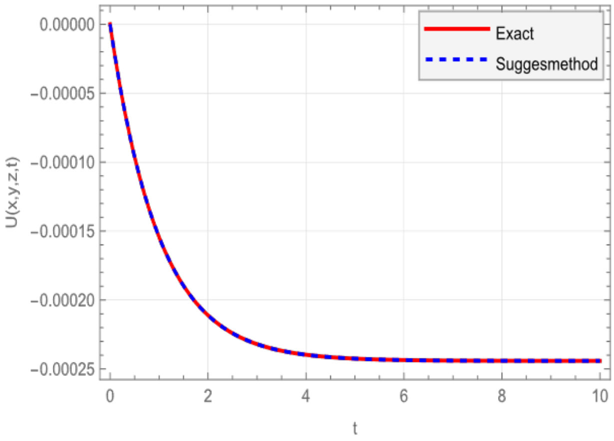

Fig. 7. Suggested Method and Exact Solution for Example 4, $\sigma = \omega = \varrho = 0.5$

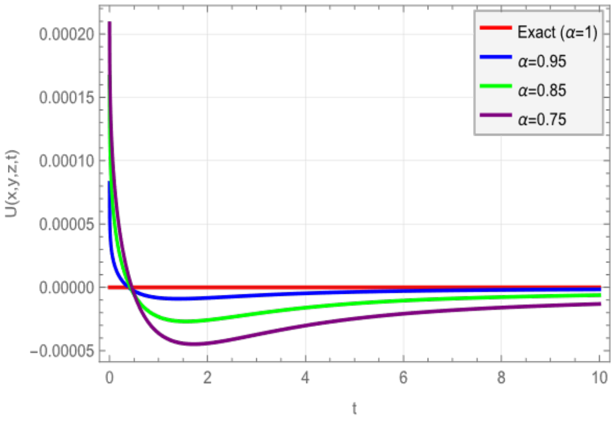

Fig. 8. Accuracy of the results, for example 4, $\sigma = \omega = \varrho = 0.5$

In Example 1, Fig. 1 shows the numerical solutions that were obtained by using the proposed method for a fractional order $\alpha = 2$. The numerical results show excellent agreement with the exact solution, the capability of the method to accurately reproduce the true behaviour of the model. Furthermore, Fig. 2 shows that absolute error always decreases with the increase of fractional order. The comparative results obtained in Table 1 comparing the performance of proposed method with the ADM, HVIM and HAM techniques reported in [30–32] clearly demonstrate the superior performance of proposed approach with regards to accuracy and stability, at $\alpha = 2$. These findings confirm the efficiency and reliability of the method to solve a fractional parabolic equation. In Example 2, Fig. 3 shows the numerical solutions calculated by the proposed scheme for the case of a fractional order $\alpha = 1$. A good agreement within the exact solution is seen with the numerical solution. As is clear from Fig. 4, the absolute error decreases as the fractional order increases. Table 2 shows a detailed comparison between the numerical results calculated for the special case $\alpha = 1$ and corresponding exact solutions. The results confirm that the proposed method gives very accurate approximations for the fractional diffusion equation. In Example 3 the numerical solutions obtained for fractional order $\alpha = 1$ are shown in Fig. 5, again showing a good accordance with the exact solution. This consistency calls attention to the capability of the proposed method to correctly capture the underlying dynamics of the problem. Besides, a clear decrease in the absolute error with the increase of the fractional order is shown in Fig. 6. The comparison in Table 3 between the proposed method and the exact solution further confirms the robustness and accuracy of the method. In Example 4, the profiles of the numerical solution obtained using the proposed technique for a fractional order $\alpha = 1$ are shown in Fig. 7. The numerical results are close to the exact solution and highlight the ability of the method to reproduce the actual behaviour of the fractional system. Fig. 8 confirms this fact that the absolute error goes down with increase in fractional order. The comparative analysis shown in Table 4 comparing the proposed approach with several methods discussed in [35] shows that the proposed method can deliver higher accuracy and better stability especially for $\alpha = 1$. The results further show the trustworthiness and effectiveness of the approach for the solution of fractional-order heat equation.

4. Conclusions

This study presents a simple and efficient analytical approach for solving time-dependent PDEs. The results demonstrate that the proposed method can provide accurate solutions with high efficiency and clear ease of implementation. The findings confirm that the method represents a reliable tool for the analysis of two- and three-dimensional fractional models. The main conclusions are summarized as follows:

- The proposed method can derive analytical solutions for time-fractional two- and threedimensional partial differential equations using the Caputo derivative, provided that suitable initial conditions are chosen.
- The method depends exclusively on the initial conditions and expresses the solution in the form of an infinite power series, without the need for additional assumptions, restrictive conditions, or perturbation techniques.
- The computational procedure is straightforward and well-structured, making the method easy to apply and characterized by low computational cost.
- The four examined examples clearly demonstrate the effectiveness and accuracy of the method, confirming its validity and practical applicability.
- For the cases $\alpha = 1$ and $\alpha = 2$, the obtained solutions show excellent agreement with the exact solutions, indicating a high level of accuracy.
- When the fractional order α is less than one, the numerical behaviour of the fractional system approaches that of the corresponding integer-order model, and this similarity increases as α approaches one.
- The results reveal that the method achieves rapid convergence, high accuracy, and minimal computational effort.
- These outcomes suggest that the proposed approach can be extended in future work to solve linear partial differential equations arising in various scientific fields, including physics, biology, medicine, and engineering.

5. Acknowledgements

We extend our thanks to Department of. Mathematics /College of Education for Pure Science (Ibn Al-Haitham)/ University of Baghdad.

References

[1] M. M. Helal, (2023) *"Double LA-Transform and Their Properties for Solving Partial Differential Equations"* **AIP Conference Proceedings *2834***(1): 080140. DOI: 10.1063/5.0163295.

[2] I. I. Gorial, (2020) *"Numerical Methods for Fractional Percolation Equation with Riesz Space Fractional Derivative"* **International Review on Modelling and Simulations *13***(6): 438–443. DOI: 10.15866/iremos.v13i6.19297.

[3] F. F. Ghazi, (2020) *"New Approach for Solving Two-Dimensional Spaces PDE"* **Journal of Physics: Conference Series *1530***(1): 012066. DOI: 10.1088/1742-6596/1530/1/012066.

[4] I. I. Gorial, (2021) *"Numerical Simulation for Fractional Percolation Equation"* **Mathematical Modelling of Engineering Problems *8***(3): 425–430. DOI: 10.18280/mmep.080312.

[5] Z. H. Kareem and L. N. M. Tawfiq, (2023) *"Recent Modification of Decomposition Method for Solving Wave-Like Equation"* **Journal of Interdisciplinary Mathematics *26***(5): 809–820. DOI: 10.47974/JIM-1235.

[6] I. I. Gorial, (2021) *"Explicit Finite Difference Approximation Method for Solving the Riesz Space Fractional Percolation Equation"* **Journal of Applied Science and Engineering *24*(6)**: 1051–1058. DOI: 10.6180/jase.202112_24(6).0020.

[7] Z. H. Kareem and L. N. M. Tawfiq, (2022) *"New Modification of Decomposition Method for Solving High Order Strongly Nonlinear Partial Differential Equations"* **AIP Conference Proceedings *2398***(1): 1–9. DOI: 10.1063/5.0094811.

[8] Z. H. Kareem, (2021) *"Efficient Modification of the Decomposition Method for Solving a System of PDEs"* **Iraqi Journal of Science *62***(9): 3061–3070. DOI: 10.24996/ijs.2021.62.9.21.

[9] I. I. Gorial, (2013) *"Analytical Solution for Two-Dimensional Fractional Dispersion Equation by Modified Decomposition Method"* **World Applied Sciences Journal 23**(8): 1049–1052. DOI: 10.5829/idosi.wasj.2013.23.08.2578.

[10] M. O. Enadi, (2019) *"New Approach for Solving Three-Dimensional Space Partial Differential Equation"* **Baghdad Science Journal *16***(3): 786–792. DOI: 10.21123/bsj.2019.16.3(Suppl.).0786.

[11] N. A. Hussein, (2022) *"Exact Solution for Systems of Nonlinear (2+1)D Differential Equations"* **Iraqi Journal of Science *63***(10): 4388–4396. DOI: 10.24996/ijs.2022.63.10.25.

[12] A. H. Khamas, (2023) *"New Approach for Calculate Exponential Integral Function"* **Iraqi Journal of Science *64***(8): 4034–4042. DOI: 10.24996/ijs.2023.64.8.27.

[13] I. I. Gorial, (2021) *"Numerical Simulation for Fractional Percolation Equation"* **Mathematical Modelling of Engineering Problems *8***(3): 425–430. DOI: 10.18280/mmep.080312.

[14] M. Tamsir and V. K. Srivastava, (2016) *"Revisiting the Approximate Analytical Solution of Fractional-Order Gas Dynamics Equation"* **Alexandria Engineering Journal *55***(2): 867–874. DOI: 10.1016/j.aej.2016.02.009.

[15] V. Daftardar-Gejji and S. Bhalekar, (2008) *"Solving Multi-Term Linear and Non-Linear Diffusion-Wave Equations of Fractional Order by Adomian Decomposition Method"* **Applied Mathematics and Computation *202***(1): 113–120. DOI: 10.1016/j.amc.2013.02.072.

[16] S. Momani and Z. Odibat, (2006) *"Analytical Solution of a Time-Fractional Navier–Stokes Equation by Adomian Decomposition Method"* **Applied Mathematics and Computation *177***(2): 488–494. DOI: 10.1016/j.amc.2005.11.025.

[17] H. Jafari and V. Daftardar-Gejji, (2006) *"Solving a System of Nonlinear Fractional Differential Equations Using Adomian Decomposition"* **Journal of Computational and Applied Mathematics *196***(2): 644–651. DOI: 10.1016/j.cam.2005.10.017.

[18] S. Momani, Z. Odibat, and V. S. Erturk, (2007) *"Generalized Differential Transform Method for Solving a Space- and Time-Fractional Diffusion-Wave Equation"* **Physics Letters A *370***(5–6): 379–387. DOI: 10.1016/j.physleta.2007.05.037.

[19] M. Kurulay, (2010) *"The Approximate and Exact Solutions of the Space- and Time-Fractional Burgers Equations"* **International Journal of Research and Review in Applied Sciences *3***(3): 257–263.

[20] S. Momani and Z. Odibat, (2008) *"A Generalized Differential Transform Method for Linear Partial Differential Equations of Fractional Order"* **Applied Mathematics Letters *21***(2): 194–199. DOI: 10.1016/j.aml.2007.02.022.

[21] N. A. Khan, A. Ara, and A. Mahmood, (2012) *"Numerical Solutions of Time-Fractional Burgers Equations: A comparison between generalized transformation technique with homotopy perturbation method"* **International Journal of Numerical Methods for Heat and Fluid Flow *24***(2): 175–193.

[22] S. Momani and Z. Odibat, (2007) *"Homotopy Perturbation Method for Nonlinear Partial Differential Equations of Fractional Order"* **Physics Letters A *365***(5–6): 345–350. DOI: 10.1016/j.physleta.2007.01.046.

[23] S. Momani and Z. Odibat, (2007) *"Comparison between the Homotopy Perturbation Method and the Variational Iteration Method for Linear Fractional Partial Differential Equations"* **Computers & Mathematics with Applications *54***(7–8): 910–919. DOI: 10.1016/j.camwa.2006.12.037.

[24] Z. Odibat and S. Momani, (2008) *"Modified Homotopy Perturbation Method Application to Quadratic Riccati Differential Equation of Fractional Order"* **Chaos, Solitons & Fractals *36***(1): 167–174. DOI: 10.1016/j.chaos.2006.06.041.

[25] X. J. Yang, D. Baleanu, Y. Khan, and S. T. Mohyud-Din, (2014) *"Local Fractional Variational Iteration Method for Diffusion and Wave Equations on Cantor Sets"* **Romanian Journal of Physics *59***(1–2): 36–48.

[26] M. Abbaszadeh, A. Bagheri Salec, and A. Jebur, (2023) *"Integrated radial basis function technique to simulate the nonlinear system of time fractional distributed-order diffusion equation with graded time-mesh discretization"* **Engineering Analysis with Boundary Elements *156***: 57–69. DOI: 10.1016/j.enganabound.2023.05.049.

[27] M. Abbaszadeh, (2019) *"Error estimate of second-order finite difference scheme for solving the Riesz space distributed-order diffusion equation"* **Applied Mathematics Letters *88***: 179–185. DOI: 10.1016/j.aml.2018.08.024.

[28] P. Karunakar and S. Chakraverty, (2019) *"Shifted Chebyshev Polynomials-Based Solution of Partial Differential Equations"* **SN Applied Sciences *1***: 285. DOI: 10.1007/s42452-019-0292-z.

[29] I. Podlubny. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*. ***198***. Mathematics in Science and Engineering. San Diego, CA, USA: Academic Press, 1998, 159–198.

[30] X. Wang, (2023) *"Approximate Solutions of Time-Fractional Fourth-Order Differential Equations with Variable Coefficients"* **Journal of Applied Science and Engineering *26***(9): 1223–1235. DOI: 10.6180/jase.202309_26(9).0003.

[31] A. Khaliq and E. Twizell, (1987) *"A Family of Second Order Methods for Variable Coefficient Fourth Order Parabolic Partial Differential Equations"* **International Journal of Computer Mathematics *23***(1): 63–76. DOI: 10.1080/00207168708803608.

[32] C. Andrade and S. McKee, (1977) *"High Accuracy A.D.I. Methods for Fourth Order Parabolic Equations with Variable Coefficients"* **Journal of Computational and Applied Mathematics *3***(1): 11–14. DOI: 10.1016/0771-050X(77)90019-5.

[33] N. Ali Shah, S. Saleem, A. Akgül, K. Nonlaopon, and J. D. Chung, (2021) *"Numerical Analysis of Time-Fractional Diffusion Equations via a Novel Approach"* **Journal of Function Spaces *2021***: 9945364. DOI: 10.1155/2021/9945364.

[34] D. Baleanu and H. K. Jassim, (2020) *"Exact Solution of Two-Dimensional Fractional Partial Differential Equations"* **Fractal and Fractional *4***(2): 21. DOI: 10.3390/fractalfract4020021.

[35] M. R. Gadallah and H. Eltayeb, (2023) *"Solutions of Time Fractional (1+3)-Dimensional Partial Differential Equations by the Natural Transform Decomposition Method"* **Axioms *12***: 958. DOI: 10.3390/axioms12100958.