Exact Solutions Of The Conformable Fractional Differential Systems With Constant Coefficients

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Conformable fractional calculus simplifies research and application due to its similarity with classical calculus in theory and operations. Therefore, it has gained significant attention from researchers and has accumulated rich theoretical achievements. The article establishes a comprehensive eigenvalue-based analytical system for constant coefficient flexible fractional differential systems by extending previous research work. The proposed method provides a systematic approach to eigenvalue classification, which results in four distinct eigenvalue categories and produces actual solutions for every category. The process of transforming complex eigenvalues into real solutions employs Euler's formula, whereas the method for handling repeated roots uses a systematic approach of variable substitution. The framework establishes a complete theoretical foundation that describes high-dimensional conformable systems and demonstrates its validity through practical examples.

Keywords: Conformable Differential Systems, Constant Coefficients, Eigenvalue.

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1. Introduction

Fractional calculus extends derivatives and integrals to non-integer orders, effectively modeling systems with memory and non-local behavior. The piecewise constant argument (PCA) method is a widely used discretization technique for conformable fractional-order differential equations, where the continuous system is approximated over small subintervals to obtain an equivalent discrete dynamical model [1, 2]. This approach facilitates stability, bifurcation, and long-term behavior analysis while preserving key properties of the conformable derivative. Several studies have demonstrated its effectiveness in modeling population dynamics, predator-prey systems, and tumor growth models through discretization and dynamical analysis [3]. However, existing PCA-based approaches primarily emphasize numerical approximation and qualitative dynamical behavior rather than deriving closed-form analytical solutions [4]. In contrast, the present work develops a systematic eigenvalue-based analytical framework that provides explicit exact solutions for conformable fractional differential systems with constant coefficients, thereby complementing discretization-based methods [5].

In 2014, Kareem et al. introduced Conformable fractional derivatives, overcoming traditional derivative limitations while retaining classical calculus properties [6]. These derivatives combine fractional models with classical system analysis, offering advantages like linearity and supporting the Differential Mean Value Theorem [7–12]. This research introduces a new framework to overcome limitations in high-order, strongly coupled nonlinear systems.

Fractional derivatives like Riemann-Liouville and Caputo help model memory and nonlocal effects, but their singular integral kernels create analytical difficulties. The operator-theoretic approaches establish rigorous results, but they need advanced functional analysis which increases the difficulty of both solution methods and computational work for constantcoefficient systems. The conformable fractional derivative preserves essential calculus properties, enabling researchers to apply traditional eigenvalue analysis methods to study fractional systems. Several existing studies focus on numerical approximation methods and qualitative analysis of conformable fractional systems; limited attention has been given to developing a unified analytical framework for obtaining exact solutions of high-dimensional systems with different eigenvalue structures. In contrast, the present study introduces a systematic eigenvalue-based approach that provides explicit general solutions for all possible eigenvalue cases, including repeated and complex roots, thereby offering a more comprehensive and structured analytical methodology.

The approach has demonstrated value across various scientific domains, yet existing structured solution techniques for high-dimensional systems with repeated or complex eigenvalues remain insufficient, thereby driving researchers to develop new generalized eigenvalue-based approaches.

Fractional-order derivatives function as accurate modeling instruments that describe biological systems which display both memory capabilities and hereditary biological processes, which include neural transmission, tumor growth and drug diffusion and epidemic spread. The conformable fractional derivative preserves conventional calculus properties while it shows memory behavior, which enables scientists to achieve higher accuracy through this mathematical tool, which assists them in studying complex biological systems.

This study proposes a data-driven framework to discover fractional differential equations directly from data. It integrates deep neural networks, automatic differentiation, Gauss-Jacobi quadrature, sparse regression, and global optimization. Validated on synthetic and experimental datasets, the method accurately identifies fractional structures and models memory effects across diverse noise conditions [13].

This study analyses the intrinsic properties of the conformable fractional derivative and introduces a semi domain soliton. It highlights differences from the Riemann-Liouville derivative, including absence of memory effects, shows transformation equivalence with integer-order equations, and establishes solution-replacement theorems, revealing distinct dynamical behaviours and modelling potential [14].

The researchers studied the existence and stability conditions of periodic systems which use non-instantaneous impulses to drive their behaviour. The study presents the conformable Cauchy matrix which researchers use to study exponential stability and derive solutions of linear nonhomogeneous systems through variation of constants. The researchers established specific conditions which guarantee that periodic solutions will exist and be unique in nonlinear systems [15].

The researchers developed a new algorithmic solution to create non-uniqueness counterexamples which demonstrate the behaviour of linear evolution partial differential equations in quarter-plane regions that have constant coefficients. The study uses complex analysis together with the Fokas unified transform to investigate boundary regularity while testing their method on both the heat equation and linear KdV equations to produce new results for the problem of uniqueness [16, 17].

This distinction highlights the novelty of the proposed approach, which not only generalizes existing solution techniques but also provides a complete classification of solution structures based on eigenvalue characteristics.

The key contributions of this paper are:

- Developed a generalized method for solving conformable fractional differential systems.
- Introduced an eigenvalue-based framework for high-dimensional, coupled systems.

2. materials and methods

This section presents the fundamental concepts and preliminary results required for the development of the proposed analytical framework. These definitions and lemmas form the theoretical basis for deriving the main results in the subsequent section.

Definition 1 [5]. The function $w : [0, \infty) \to R$ is differentiable, then

$$D_\theta w(\tau) = \lim_{\rho \to 0} \frac{w\left(\tau + \rho\tau^{1-\theta}\right) - w(\tau)}{\rho}, \theta \in (0,1]. \tag{1}$$

Definition 2 [5].

$$I_\theta w(v) = \int v^{\theta-1} w(v) dv, 0 < \theta \leq 1. \tag{2}$$

Lemma 1 [5]. Assume $w'(\tau)$ exists, then we have $D_\theta I_\theta w(\tau) = w(\tau), \theta \in (0,1]$.

Lemma 2 [5]. Let $\theta \in (0,1]$, If the functions $u(\tau)$ and $v(\tau)$ are differentiable, then

$$D_\theta(mu + nv) = mD_\theta u + nD_\theta v \tag{3}$$

$$D_\theta(uv) = uD_\theta v + vD_\theta u \tag{4}$$

$$D_\theta\left(\frac{u}{v}\right) = \frac{vD_\theta u - uD_\theta v}{v^2} \tag{5}$$

$$D_\theta u(\tau) = \tau^{1-\theta}\frac{du(\tau)}{d\tau} \tag{6}$$

Lemma 3.

If the functions $y_1(t), y_2(t), \ldots, y_n(t)$ are linearly independent on an interval I, then their Wronskian determinant

$$W(t) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \tag{7}$$

is nonzero for at least one $t \in I$.

Proof

The above definitions and lemmas provide the necessary mathematical foundation for analyzing conformable fractional systems. In particular, the concept of linear independence and properties of the conformable derivative play a crucial role in constructing the general solutions presented in the next section.

Assume a linear combination of the functions equals zero:

$$c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) = 0 \tag{8}$$

for all $t \in I$.

Differentiating this equation $n-1$ times gives the system:

$$\sum_{i=1}^{n} c_i y_i^{(k)}(t) = 0, k = 0, 1, \ldots, n-1 \tag{9}$$

This can be written in matrix form as

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{10}$$

If the Wronskian determinant $W(t) \neq 0$, the coefficient matrix is nonsingular, and the only solution is

$$c_1 = c_2 = \cdots = c_n = 0 \tag{11}$$

Hence, the functions are linearly independent.

Conversely, if the functions are linearly independent, the above system admits only the trivial solution, implying that the determinant must be nonzero for some $t \in I$.

Therefore, the functions are linearly independent if and only if the Wronskian determinant is nonzero.

2. Main results

This section develops a systematic eigenvalue-based approach for solving conformable fractional differential systems. The solutions are categorized according to the nature and multiplicity of eigenvalues to ensure a structured and comprehensive analysis.

The exact solutions of conformable fractional differential systems with constant coefficients are derived in this section. The results are classified according to the nature of the eigenvalues of the associated characteristic equation, which includes single real roots and complex roots and repeated real roots and repeated complex roots.

To improve clarity and mathematical rigor, each result in this section is derived through explicit intermediate steps. In particular, all assumed solution forms are substituted into the original conformable fractional system, and the validity of transformations is justified using properties of the conformable derivative. Linear independence of solutions is also verified using the Wronskian determinant where necessary.

Theorem 1. Consider the following conformable fractional differential system with constant coefficients, where $\theta \in (0, 1], b_i (i = 1, 2, \ldots, n)$ are constant coefficients,

$$D_\theta^{(n)} w(\tau) + b_1 D_\theta^{(n-1)} w(\tau) + \cdots + b_{n-1} D_\theta w(\tau) + b_n w(\tau) = 0 \tag{12}$$

if there are only single real-root eigenvalues to the characteristic equation,

$$p^n + b_1p^{n-1} + \ldots + b_{n-2}p^2 + b_{n-1}p + b_n = 0 \tag{13}$$

then the general solution to the conformable fractional differential system (11) is:

$$w(\tau) = c_1e^{p_1\frac{\tau^\theta}{\theta}} + c_2e^{p_2\frac{\tau^\theta}{\theta}} + \ldots + c_{n-1}e^{p_{n-1}\frac{\tau^\theta}{\theta}} + c_ne^{p_n\frac{\tau^\theta}{\theta}} \tag{14}$$

Proof. The general solution to the equation $D_\theta w(\tau) + bw(\tau) = 0$ is $w(\tau) = ce^{-b\frac{\tau^\theta}{\theta}}$. Thus, we propose that the solution to equation (12) is of the form $w(\tau) = e^{p\frac{\tau^\theta}{\theta}}$. Substituting this trial solution into (12) yields:

$$D_\theta^{(n)}\left(e^{p\frac{\tau^\theta}{\theta}}\right) = p^ne^{p\frac{\tau^\theta}{\theta}} \tag{15}$$

Substituting into equation (12) gives:

$$\left(p^n + b_1 p^{n-1} + \ldots + b_{n-2} p^2 + b_{n-1} p + b_n\right) e^{p\frac{\tau^\theta}{\theta}} = 0 \tag{16}$$

Since $e^{p\frac{\tau^\theta}{\theta}} \neq 0$, it follows that the characteristic equation is satisfied:

$$p^n + b_1 p^{n-1} + \ldots + b_{n-2} p^2 + b_{n-1} p + b_n = 0. \tag{17}$$

The real roots $p_1, p_2, \ldots, p_n$ of this equation provide the linearly independent solutions $e^{p_1\frac{\tau^\theta}{\theta}}, e^{p_2\frac{\tau^\theta}{\theta}}, \ldots, e^{p_{n-1}\frac{\tau^\theta}{\theta}}, e^{p_n\frac{\tau^\theta}{\theta}}$. The Wronskian determinant can be shown to be nonzero, confirming the linear independence of the solutions. Thus,

$$w(\tau) = c_1 e^{p_1 \tau^\theta/\theta} + c_2 e^{p_2 \tau^\theta/\theta} + \cdots + c_n e^{p_n \tau^\theta/\theta} \tag{18}$$

Theorem 2. Consider the following conformable fractional differential system with constant coefficients, where $\theta \in (0,1], b_i (i = 1, 2, \ldots, n)$ are constant coefficients,

$$\begin{aligned} D_\theta^{(n)} w(\tau) + b_1 D_\theta^{(n-1)} w(\tau) + \cdots + b_{n-2} D_\theta^{(2)} w(\tau) \\ + b_{n-1} D_\theta w(\tau) + b_n w(\tau) = 0 \end{aligned} \tag{19}$$

if $p = a + ib$ is the complex-root eigenvalue to the characteristic equation,

$$p^n + b_1 p^{n-1} + \ldots + b_{n-2} p^2 + b_{n-1} p + b_n = 0 \tag{20}$$

then the conformable fractional differential system (18) has the real solutions:

$$e^{a\frac{\tau^\theta}{\theta}} \cos\left(b\frac{\tau^\theta}{\theta}\right), e^{a\frac{\tau^\theta}{\theta}} \sin\left(b\frac{\tau^\theta}{\theta}\right) \tag{21}$$

Proof. We use Euler's formula to obtain the solution:

$$w(\tau) = e^{(a+ib)\frac{\tau^\theta}{\theta}} = e^{a\frac{\tau^\theta}{\theta}} e^{ib\frac{\tau^\theta}{\theta}} = e^{a\frac{\tau^\theta}{\theta}} \cos\left(b\frac{\tau^\theta}{\theta}\right) + ie^{a\frac{\tau^\theta}{\theta}} \sin\left(b\frac{\tau^\theta}{\theta}\right) \tag{22}$$

$$\varphi(\tau) = e^{a\frac{\tau^\theta}{\theta}} \cos\left(b\frac{\tau^\theta}{\theta}\right), \quad \phi(\tau) = e^{a\frac{\tau^\theta}{\theta}} \sin\left(b\frac{\tau^\theta}{\theta}\right) \tag{23}$$

Substituting $w(\tau) = \varphi(\tau) + i\phi(\tau)$ into (18) gives:

$$D_\theta^{(n)} \varphi(\tau) + b_1 D_\theta^{(n-1)} \varphi(\tau) + \ldots + b_{n-2} D_\theta^{(2)} \varphi(\tau) + b_{n-1} D_\theta \varphi(\tau) + b_n \varphi(\tau) = 0 \tag{24}$$

$$D_\theta^{(n)} \phi(\tau) + b_1 D_\theta^{(n-1)} \phi(\tau) + \ldots + b_{n-2} D_\theta^{(2)} \phi(\tau) + b_{n-1} D_\theta \phi(\tau) + b_n \phi(\tau) = 0 \tag{25}$$

Thus, $\varphi(\tau) = e^{a\frac{\tau^\theta}{\theta}} \cos\left(b\frac{\tau^\theta}{\theta}\right)$ and $\phi(\tau) = e^{a\frac{\tau^\theta}{\theta}} \sin\left(b\frac{\tau^\theta}{\theta}\right)$ are the real solutions. Furthermore, the obtained real-valued solutions are verified by direct substitution into the original system (19), confirming that they satisfy the conformable fractional differential equation.

Theorem 3. Consider the following conformable fractional differential system with constant coefficients, where $\theta \in (0,1], b_i (i = 1, 2, \ldots, n)$ are constant coefficients,

$$D_\theta^{(n)} w(\tau) + b_1 D_{theta}^{(n-1)} w(\tau) + \ldots + b_{n-2} D_{theta}^{(2)} w(\tau) + b_{n-1} D_{theta} w(\tau) + b_n w(\tau) = 0 \tag{26}$$

if $p = p_1$ is the k -fold real-root eigenvalue to the characteristic equation:

$$p^n + b_1 p^{n-1} + \ldots + b_{n-2} p^2 + b_{n-1} p + b_n = 0 \tag{27}$$

Then the conformable fractional differential system (26) has the real solutions:

$$e^{p_1\frac{\tau^\theta}{\theta}}, \frac{\tau^\theta}{\theta} e^{p_1\frac{\tau^\theta}{\theta}}, \ldots \left(\frac{\tau^\theta}{\theta}\right)^2 e^{p_1\frac{\tau^\theta}{\theta}}, \ldots, \left(\frac{\tau^\theta}{\theta}\right)^{k-1} e^{p_1\frac{\tau^\theta}{\theta}}. \tag{28}$$

Proof. of our proof requires two separate cases because the foremost coefficient p_1 determines the system's structure for conformable fractional systems.

Case 1 , if $p_1 = 0$, the characteristic equation takes the form:

$$p^n + b_1 p^{n-1} + \ldots + b_{n-k} p^k = 0, \tag{29}$$

and the corresponding conformable fractional differential system is:

$$D_\theta^{(n)} w(\tau) + b_1 D_\theta^{(n-1)} w(\tau) + \ldots + b_{n-k} D_\theta^{(k)} w(\tau) = 0. \tag{30}$$

According to the definition of conformable fractional calculus, it is obvious that $1, : \frac{\tau^\theta}{\theta}, \left(\frac{\tau^\theta}{\theta}\right)^2, \ldots, \left(\frac{\tau^\theta}{\theta}\right)^{k-1}$ are the solutions of the above differential system.

Case 2, if $p_1 \neq 0$, let $u(\tau) = v(\tau) e^{p_1\frac{\tau^\theta}{\theta}}$, According to Lemma 2, we introduce the transformation x = t^α to reduce the conformable fractional system to an equivalent classical differential equation.

$$T_\alpha(f(t)) = t^{1-\alpha} \frac{df}{dt} \tag{31}$$

which ensures equivalence between the conformable system and the transformed integer-order system. After solving the transformed system, the inverse substitution is applied, and each resulting solution is substituted back into the original equation (26) to verify correctness.

The second lemma enables us to obtain the following result:

$$\begin{aligned} D_\theta u(\tau) &= D_\theta \left[v(\tau)e^{p_1\tau^\theta/\theta}\right] \\ &= p_1 v(\tau)e^{p_1\tau^\theta/\theta} + e^{p_1\tau^\theta/\theta}D_\theta v(\tau) \end{aligned} \tag{32}$$

$$D_\theta^{(2)}u(\tau) = e^{p_1\tau^\theta/\theta}\left[D_\theta^{(2)}v(\tau) + 2p_1 D_\theta v(\tau) + p_1^2 v(\tau)\right] \tag{33}$$

$$\begin{aligned} D_\theta^{(m)}u(\tau) = e^{p_1\tau^\theta/\theta}\Big[D_\theta^{(m)}v(\tau) + mp_1 D_\theta^{(m-1)}v(\tau) \\ + \cdots + p_1^m v(\tau)\Big] \end{aligned} \tag{34}$$

Substituting the above expression into (26), we can obtain $c_i(i = 1, 2, \ldots, n)$,

$$\begin{aligned} D_\theta^{(n)}u(\tau) + c_1 D_\theta^{(n-1)}u(\tau) + \ldots \\ + c_{n-1}D_\theta u(\tau) + c_n u(\tau) = 0 \end{aligned} \tag{35}$$

This corresponds to the characteristic equation:

$$\lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-2}\lambda^2 + c_{n-1}\lambda + c_n = 0 \tag{36}$$

$$\left[(p_1+\lambda)^n + b_1(p_1+\lambda)^{n-1} + \cdots + b_{n-1}(p_1+\lambda) + b_n\right] e^{(p_1+\lambda)\tau^\theta/\theta} = 0 \tag{37}$$

Since $e^{\frac{(p_1+\lambda)\tau^\theta}{\theta}} \neq 0$, we have

$$\left[(p_1+\lambda)^n + b_1(p_1+\lambda)^{n-1} + \cdots + b_{n-2}(p_1+\lambda)^2 + b_{n-1}(p_1+\lambda) + b_n\right] \tag{38}$$

$$\left[\lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-2}\lambda^2 + c_{n-1}\lambda + c_n\right] \tag{39}$$

The eigenvalue $p = p_1 \neq 0$ of (26) corresponds to the eigenvalue $\lambda = 0$ of (34). If $\lambda = 0$ is the k -fold real-root eigenvalue to the characteristic equation (35), then,

Therefore, the transformed system (35) has a k -fold real eigenvalue λ. According to the classical theory of linear differential equations with repeated eigenvalues, the corresponding linearly independent solutions take the form

$$e^{\lambda x}, xe^{\lambda x}, x^2 e^{\lambda x}, \ldots, x^{k-1}e^{\lambda x} \tag{40}$$

Substituting the transformation $x = t^\alpha$ back into these expressions' yields

$$e^{\lambda t^\alpha}, t^\alpha e^{\lambda t^\alpha}, t^{2\alpha}e^{\lambda t^\alpha}, \ldots, t^{(k-1)\alpha}e^{\lambda t^\alpha} \tag{41}$$

which correspond exactly to the solution forms given in equation (28). Hence, the functions listed in (27) satisfy the conformable fractional differential system (26), completing the proof.

$$\frac{\tau^\theta}{\theta}, \left(\frac{\tau^\theta}{\theta}\right)^2, \ldots, \left(\frac{\tau^\theta}{\theta}\right)^{k-1} \tag{42}$$

$$e^{p_1\frac{\tau^\theta}{\theta}}, \frac{\tau^\theta}{\theta}e^{p_1\frac{\tau^\theta}{\theta}}, \left(\frac{\tau^\theta}{\theta}\right)^2 e^{p_1\frac{\tau^\theta}{\theta}}, \ldots, \left(\frac{\tau^\theta}{\theta}\right)^{k-1} e^{p_1\frac{\tau^\theta}{\theta}} \tag{43}$$

Finally, each obtained solution is substituted back into the original conformable system (26) to verify that it satisfies the equation, thereby completing the proof with full mathematical justification.

Theorem 4. Consider the following conformable fractional differential system with constant coefficients, where $\theta \in (0, 1], b_i(i = 1, 2, \ldots, n)$ are constant coefficients,

$$D_\theta^{(n)}w(\tau) + b_1 D_\theta^{(n-1)}w(\tau) + \ldots + b_{n-2}D_\theta^{(2)}w(\tau) + b_{n-1}D_\theta w(\tau) + b_n w(\tau) = 0 \tag{44}$$

if $p = \alpha + i\beta$ is the k -fold complex-root eigenvalue to the characteristic equation,

$$p^n + b_1 p^{n-1} + \ldots + b_{n-2}p^2 + b_{n-1}p + b_n = 0, \tag{45}$$

Then the conformable fractional differential system (41) has the real solutions:

$$e^{\alpha\frac{\tau^\theta}{\theta}}\cos\left(\beta\frac{\tau^\theta}{\theta}\right), \frac{\tau^\theta}{\theta}e^{\alpha\frac{\tau^\theta}{\theta}}\cos\left(\beta\frac{\tau^\theta}{\theta}\right), \ldots, \left(\frac{\tau^\theta}{\theta}\right)^{k-1}e^{\alpha\frac{\tau^\theta}{\theta}}\cos\left(\beta\frac{\tau^\theta}{\theta}\right) \tag{46}$$

$$e^{\alpha\frac{\tau^\theta}{\theta}}\sin\left(\beta\frac{\tau^\theta}{\theta}\right), \frac{\tau^\theta}{\theta}e^{\alpha\frac{\tau^\theta}{\theta}}\sin\left(\beta\frac{\tau^\theta}{\theta}\right), \ldots, \left(\frac{\tau^\theta}{\theta}\right)^{k-1}e^{\alpha\frac{\tau^\theta}{\theta}}\sin\left(\beta\frac{\tau^\theta}{\theta}\right) \tag{47}$$

Proof. By using the method in Theorem 3, we can obtain the following solutions to (42):

$$e^{(\alpha+i\beta)\frac{\tau^\theta}{\theta}}, \frac{\tau^\theta}{\theta}e^{(\alpha+i\beta)\frac{\tau^\theta}{\theta}}, \left(\frac{\tau^\theta}{\theta}\right)^2 e^{(\alpha+i\beta)\frac{\tau^\theta}{\theta}}, \ldots, \left(\frac{\tau^\theta}{\theta}\right)^{k-1} e^{(\alpha+i\beta)\frac{\tau^\theta}{\theta}} \tag{48}$$

Using Euler's formula, we can rewrite the complex exponential as:

$$e^{(\alpha+i\beta)\frac{\tau^\theta}{\theta}} = e^{\alpha\frac{\tau^\theta}{\theta}}\left[\cos\left(\beta\frac{\tau^\theta}{\theta}\right) + i\sin\left(\beta\frac{\tau^\theta}{\theta}\right)\right] \tag{49}$$

Thus,

$$e^{\alpha\frac{\tau^\theta}{\theta}}\cos\left(\beta\frac{\tau^\theta}{\theta}\right), \frac{\tau^\theta}{\theta}e^{\alpha\frac{\tau^\theta}{\theta}}\cos\left(\beta\frac{\tau^\theta}{\theta}\right), \ldots, \left(\frac{\tau^\theta}{\theta}\right)^{k-1}e^{\alpha\frac{\tau^\theta}{\theta}}\cos\left(\beta\frac{\tau^\theta}{\theta}\right) \tag{50}$$

$$e^{\alpha\frac{\tau^\theta}{\theta}}\sin\left(\beta\frac{\tau^\theta}{\theta}\right), \frac{\tau^\theta}{\theta}e^{\alpha\frac{\tau^\theta}{\theta}}\sin\left(\beta\frac{\tau^\theta}{\theta}\right), \ldots, \left(\frac{\tau^\theta}{\theta}\right)^{k-1}e^{\alpha\frac{\tau^\theta}{\theta}}\sin\left(\beta\frac{\tau^\theta}{\theta}\right) \tag{51}$$

The obtained real-valued solutions are verified by substitution into the original system (42), confirming that they satisfy the conformable fractional differential equation.

3. Example applications

This section demonstrates the practical implementation of the theoretical results derived in Section 3. Each example is selected to correspond to a specific eigenvalue case, thereby illustrating the applicability and effectiveness of the proposed framework. Each example specifies eigenvalue type and multiplicity from the characteristic equation, then applies the corresponding Section 3 theorem. The solution process for problems maintains systematic progress because eigenvalues determine solution paths, which enhances both the clarity and logical structure of the solution.

Example 1. Solve the general solution to the conformable fractional differential system:

$$D_\theta^{(4)} w(\tau) - w(\tau) = 0 \tag{52}$$

Solution: The characteristic equation to the above system is

$$p^4 - 1 = 0. \tag{53}$$

The eigenvalues are $p = 1, -1, i, -i$ and the general solution is derived from Theorem 1&2.

$$w(\tau) = a_1 e^{\frac{\tau^\theta}{\theta}} + a_2 e^{\frac{-\tau^\theta}{\theta}} + a_3 \cos\frac{\tau^\theta}{\theta} + a_4 \sin\frac{\tau^\theta}{\theta}. \tag{54}$$

where $a_i (i = 1, 2, 3, 4)$ are constant coefficients.
Example 2. Solve the general solution to the conformable fractional differential system:

$$D_\theta^{(3)} w(\tau) + w(\tau) = 0 \tag{55}$$

Solution: The characteristic equation to the above system is

$$p^3 + 1 = 0 \tag{56}$$

The eigenvalues are $p = -1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}$ and the solution is derived from Theorem 1&2.

$$w(\tau) = a_1 e^{-\frac{\tau^\theta}{\theta}} + a_2 e^{\frac{\tau^\theta}{2\theta}} \cos\left(\frac{\sqrt{3}}{2}\frac{\tau^\theta}{\theta}\right) + a_3 e^{\frac{\tau^\theta}{2\theta}} \sin\left(\frac{\sqrt{3}}{2}\frac{\tau^\theta}{\theta}\right), \tag{57}$$

Example 3. Solve the general solution to the conformable fractional differential system

$$D_\theta^{(3)} w(\tau) - 3D_\theta^{(2)} w(\tau) + 3D_\theta w(\tau) - w(\tau) = 0 \tag{58}$$

Solution: The characteristic equation to the above system is

$$p^3 - 3p^2 + 3p - 1 = 0 \tag{59}$$

$p = 1$ is the 3-fold real-root eigenvalue, and the general solution is derived from Theorem 3.

$$w(\tau) = a_1 e^{\frac{\tau^\theta}{\theta}} + a_2 \frac{\tau^\theta}{\theta} e^{\frac{\tau^\theta}{\theta}} + a_3 \left(\frac{\tau^\theta}{\theta}\right)^2 e^{\frac{\tau^\theta}{\theta}} \tag{60}$$

Example 4. Solve the general solution to the conformable fractional differential system

$$D_\theta^{(4)} w(\tau) + 2D_\theta^{(2)} w(\tau) + w(\tau) = 0 \tag{61}$$

Solution: The characteristic equation to the above system is

$$p^4 + 2p^2 + 1 = 0 \tag{62}$$

The complex roots $p = i$ and $p = -i$ each with multiplicity 2 , are repeated roots, and the general solution is derived from Theorem 4.

$$w(\tau) = a_1 \cos\left(\frac{\tau^\theta}{\theta}\right) + a_2 \frac{\tau^\theta}{\theta} \cos\left(\frac{\tau^\theta}{\theta}\right) + a_3 \sin\left(\frac{\tau^\theta}{\theta}\right) + a_4 \frac{\tau^\theta}{\theta} \sin\left(\frac{\tau^\theta}{\theta}\right) \tag{63}$$

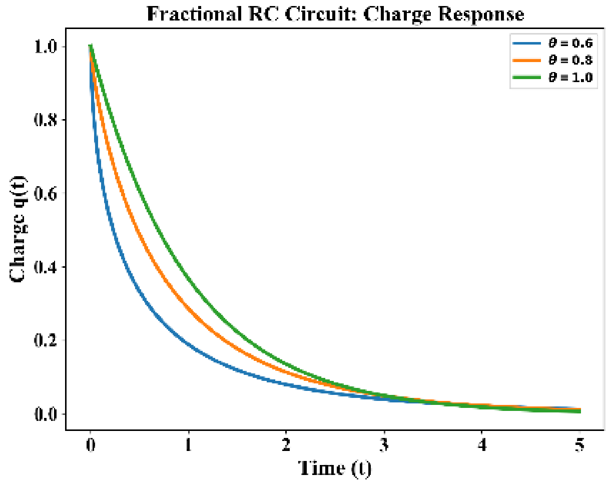

Fig. 1. The charge response $q(t)$ of a fractional RC circuit for $\theta = 0.6, 0.8, 1.0$

Fig. 1, Fig. 2 illustrates the dynamic behavior of a fractional RC circuit. (Fig. 1) presents the charge response $q(t)$ for fractional orders $\theta = 0.6, 0.8, 1.0$, while subfigure (Fig. 2) shows the corresponding current response $i(t)$ for the same fractional orders show as tablel.

For better clarity and comparison of solution structures, the general forms corresponding to each eigenvalue case are summarized in Table 1.

Table 1. General Solution Forms Based on Eigenvalue Types for Conformable Fractional Systems

Case	Eigenvalue Type	Multiplicity	General Solution Form
1	Distinct real eigenvalues λ	1	$y(t) = C\,e^{\lambda t^{\alpha}}$
2	Complex conjugate eigenvalues $\lambda = a \pm ib$	1	$y(t) = e^{at^{\alpha}}(C_1\cos(bt^{\alpha}) + C_2\sin(bt^{\alpha}))$
3	Repeated real eigenvalue λ	$k \geq 2$	$y(t) = e^{\lambda t^{\alpha}}\left(C_1 + C_2t^{\alpha} + \cdots + C_k(t^{\alpha})^{k-1}\right)$
4	Repeated complex eigenvalues $\lambda = a \pm ib$	$k \geq 2$	$y(t) = e^{at^{\alpha}}(P_{k-1}(t^{\alpha})\cos(bt^{\alpha}) + Q_{k-1}(t^{\alpha})\sin(bt^{\alpha}))$

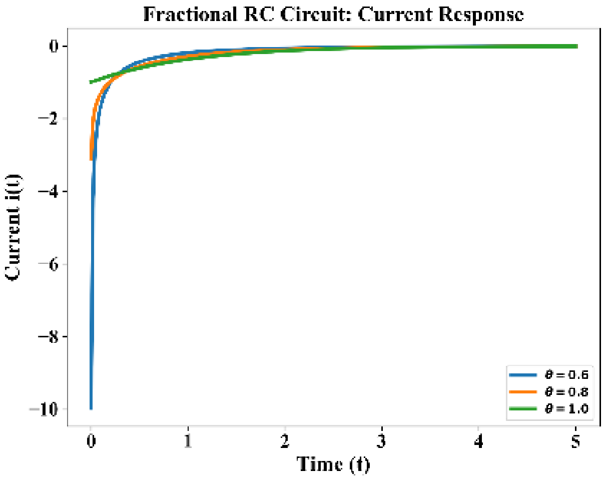

Fig. 2. The current response $i(t)$ for the same fractional orders

Table 1 provides a structured comparison of solution forms based on eigenvalue types and multiplicities, which enhances the interpretability of the theoretical results. The solutions of the problem depend on two factors which are the eigenvalue type and its corresponding eigenvalue multiplicity. The real roots produce exponential functions whereas the complex roots create oscillatory components and the repeated roots create polynomial factors which multiply both exponential and trigonometric functions.

4. Conclusions

This study developed a systematic eigenvalue-based analytical framework for solving conformable fractional differential systems with constant coefficients. The proposed approach provides explicit general solutions for different eigenvalue cases, including distinct real roots, complex roots, repeated real roots, and repeated complex roots. The classification of solutions based on eigenvalue type improves the understanding of the structural behavior of conformable systems and ensures a unified solution methodology.

The results demonstrate that the conformable fractional derivative enables the extension of classical eigenvalue techniques to fractional systems while maintaining analytical tractability. The effectiveness of the approach is validated through multiple illustrative examples corresponding to different eigenvalue structures.

Future research can focus on extending this framework to nonlinear conformable fractional systems, variable coefficient models, and higher-dimensional coupled systems. Additionally, the integration of numerical techniques with the proposed analytical method may further enhance its applicability to real-world complex systems.

Future work will focus on exploring more complex systems with higher-dimensional eigenvalue problems.

5. Declarations

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Conflict of Interest: The authors declare that they have no conflicts of interest regarding this work.

Data Availability: The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Code Availability: Not applicable.

Clinical Trial Registration: We have not harmed any human person with our research data collection, which was gathered from an already published article

Consent To Publish Declaration: Not applicable.

Consent To Participate Declaration: Not applicable.

Ethics Declaration: Not applicable.

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