Strong Fenchel Duality For Evenly Convex Optimization Problems

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Among a variety of approaches introduced in the literature to establish duality theory, Fenchel duality was of great importance in convex analysis and optimization. In this paper we establish some conditions to obtain classical strong Fenchel duality for evenly convex optimization problems defined in infinite dimensional spaces. The objective function of the primal problem is a family of (possible) infinite even convex functions. The strong duality conditions we present are based on the consideration of the epigraphs of the c-conjugate of the dual objective functions and the ε-c-subdifferential of the primal objective functions.

Keywords: evenly convex set and function, c-conjugate function, ε - c-subdifferentiability of a function, Fenchel duality.

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1. Introduction

Even convex sets (for short, e-convex sets) was first introduced by Fenchel in 1952 [1], in finite dimensional space, as the intersection of arbitrary family of open half spaces. This class of sets is very broad which may include, in addition to open convex sets, closed convex sets and other sets that are neither open nor closed. For more detailed studies of e-convex sets, see [2–4]. In [5], e-convex functions are defined as those functions whose epigraphs are e-convex sets.

The study of e-convex sets and e-convex functions motivate the researchers to generalize some concepts from classical convex analysis to this more general framework. For instance, in [6], the support function for e-convex sets, the generalized conjugate functions of e-convex functions which is called c-conjugate functions, and $\varepsilon - c$-subdifferential of a function which extends the ε-subdifferential for lower semi continuous convex functions are presented. The proceeding concepts are applied to generalize some classical fundamental results from convex analysis and optimization problems to the framework of evenly convex analysis [7].

Motivated by these results, we consider infinite dimensional optimization problems for which the cost function is the sum of infinitely even convex and proper functions. We obtain strong Fenchel duality using the classical approach for this problem. To do this, we extend some well-known results related to the epigraph of the sum of infinitely conjugate functions and the ϵ-subdifferential of proper lower semi continuous convex functions to the framework of even convex analysis. Then, we used these results to obtain strong Fenchel duality for the problem (P)

$$\min \sum_{i\in\Omega} f_i(x)$$

subject to $x \in X$

where X is real Banach space, $f(x) = \sum_{i\in\Omega} f_i(x)$ and $\{f, f_i\}_{i\in\Omega}$ is a family of infinitely many e-convex and proper functions with $f, f_i : X \to R \cup \{+\infty\}$ for all $i \in \Omega$.

In section 2, we recall some needed definitions and preliminary results related to e-convex analysis, especially, c-conjugation scheme, e-affine functions and ϵ-c-subdifferentiability with their fundamental properties. Section 3 is devoted to developing conditions which show strong duality for a primal-dual optimization problem. The primal problem is a family of infinitely many proper e-convex functions. Throughout section 3 , some essential related properties are proved.

2. Preliminaries

Throughout this paper, the standard terminologies from convex analysis are used and some important concepts are recalled. Let X is real Banach space, the non-empty set $A \subseteq X$, and X^* is the dual space of X. Denote by R the set of real numbers, and $\bar{R} = R \cup \{\pm\infty\}$ the set of extended real numbers. Let $f : X \to \bar{R}$ be an extended real valued function. The function f is called proper if domf $= \{x \in X : f(x) < +\infty\} \neq \phi$ and $f(x) \neq -\infty$ for any $x \in X$. The epigraph of f is defined as epi$f = \{(x,\gamma) \in X \times R : f(x) \leq \gamma\}$

Definition 2.1. [1, 8] The set A is said to be e-convex if A is the intersection of arbitrary family of open half spaces.

Definition 2.2. [8] The intersection of all e-convex sets containing the set A is called e-convex hull of A, and is denoted as eco A.

Definition 2.3. [5] f is e-convex if epi f is e-convex set. Furthermore, the e-convex hull of f, denoted by eco $f : X \to \overline{\mathbb{R}}$, is defined as eco$f = \sup\{h : h$ is e-convex and $h \leq f\}$

Definition 2.4. [9] The Fenchel conjugate of f is denoted by $f^* : X^* \to \overline{\mathbb{R}}$ and is defined as $f^*(v) = \sup\{< v, x > -f(x) : x \in X\}$.

Note that, the above classical Fenchel conjugate is not appropriate for e-convex functions since different e-convex functions may have same identical second conjugates. Thus, new conjugate functions called c-conjugate and c'-conjugate functions, that are suitable for e-convex functions, are introduced in [6]. These functions are adopted from the generalized conjugation functions defined in [10] and described in [11]. The definition of c-conjugate and c'-conjugate functions are given next.

Definition 2.5. [6] Let $H = X^* \times X^* \times \mathbb{R}$. The c-conjugate function of f is denoted by $f^c : H \to \overline{\mathbb{R}}$ such that $f^c(v,w,\beta) = \sup\{c(x,(v,w,\beta)) - f(x) : x \in X\}$, where $c : X \times H \to \overline{\mathbb{R}}$ is defined as

$$c(x,(v,w,\beta)) = \begin{cases} \langle x,v\rangle, & \text{if } \langle x,w\rangle < \beta \\ +\infty, & \text{if } \langle x,w\rangle \geq \beta \end{cases}$$

From the definition of f^c, the generalized Young's inequality is obtained. Namely, for any $x \in X$ and $(v,w,\beta) \in H$, we get $f^c(v,w,\beta) + f(x) - c(x,((v,w,\beta)) \geq 0$. Similarly, the c'-conjugate function of $g : H \to \bar{R}$, is the function $g^{c'} : X \to \bar{R}$ defined as $g^{c'}(x) = \sup\{c'((v,w,\beta),x) - g(v,w,\beta) : (v,w,\beta) \in H\}$, where $c' : H \times X \to \bar{R}$ such that $c'((v,w,\beta),x) = c(x,(v,w,\beta))$

Proposition 2.6. [11] Let $f : X \to \bar{R}$ and $g : H \to \bar{R}$. Then

- f^c is e'-convex and $g^{c'}$ is e-convex.
- If $f \not\equiv -\infty$ then f is e-convex if and only if $f = f^{cc'}$
- g is e'-convex if and only if $g = g^{c'c}$.

Definition 2.7. [6] The function $h : X \to \overline{\mathbb{R}}$ such that $h(x) = c(x,(v,w,\beta)) - \gamma$ is called c-elementary function where $c(x,(v,w,\beta))$ is defined as in Definition 2.5 and $\gamma \in \mathbb{R}$. The function $h' : H \to \overline{\mathbb{R}}$ such that $h'(v,w,\beta) = c'((v,w,\beta),x) - \gamma$ is called c'-elementary function where $c'((v,w,\beta),x)$ is defined as in Definition 2.5.

The family of all c-elementary (resp., c'-elementary) functions is denoted by Φ_c (resp., $\Phi_{c'}$. Thus, $\Phi_c = \{h : h$ is c-elementary functions $\}$ and $\Phi_{c'} = \{h' : h'$ is c'-elementary functions $\}$. Moreover, a function f is called Φ_c-convex if $f = \sup\{h : X \to \bar{R} : h \in \Phi_c\}$. Likewise, a function g is called $\Phi_{c'}$-convex if $g = \sup\{h' : H \to \bar{R} : h' \in \Phi_{c'}\}$.

Remark 2.8. [7] A function $g : H \to \overline{\mathbb{R}}$ is said to be e'-convex if it is $\Phi_{c'}$-convex.

Definition 2.9. [7] A set $B \subseteq H \times \mathbb{R}$ is called e'-convex if there is e'-convex function $k : H \to \overline{\mathbb{R}}$ such that $B =$ epi k. Moreover, the e'-convex hull of an arbitrary set $B \subseteq H \times \mathbb{R}$, denoted by e' co B, is the smallest e'-convex set containing B.

The next definitions are relevant and needed.

Definition 2.10. [6] A function $a : X \to \overline{\mathbb{R}}$ is called e-affine if there exists $v, w \in X^*$ and $\beta, \gamma \in \mathbb{R}$ such that

$$a(x) = \begin{cases} \langle x,v\rangle - \gamma, & \text{if } \langle x,w\rangle < \beta \\ +\infty, & \text{otherwise.} \end{cases}$$

The set of all e-affine functions minorizing any function $f : X \to \bar{R}$ is denoted by ϵ_f and is defined as $\epsilon_f = \{a : X \to \bar{R} : a$ is e$-$ affine and $a \leq f\}$.

Theorem 2.11. [6] f is e-convex and proper functions if and only if f=sup$\left\{a : a \in \varepsilon_f\right\}$.

Definition 2.12. [7] Let $f, g : X \to \overline{\mathbb{R}}$ be any two functions. An e-affine function $a \in \tilde{\varepsilon}_{f,g}$ if there exists two e-affine functions $a_1 \in \varepsilon_f, a_2 \in \varepsilon_g$ such that for

$$a_1(x) = \begin{cases} \langle x,v_1\rangle - \gamma_1, & \text{if } \langle x,w_1\rangle < \beta_1, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$a_2(x) = \begin{cases} \langle x,v_2\rangle - \gamma_2, & \text{if } \langle x,w_2\rangle < \beta_2 \\ +\infty, & \text{otherwise} \end{cases}$$

we have

$$a(x) = \begin{cases} < x, v_1 + v_2 > -(\gamma_1 + \gamma_2), & < x, w_1 + w_2 >< \beta_1 + \beta_2, \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus, the set

$$\tilde{\varepsilon}_{f,g} = \{a : X \to \bar{R} : a \text{ is } e- \text{ affine and } a \leq a_1 + a_2 \leq f + g\} \tag{1}$$

Remark 2.13. [7] From [1], we have: $\tilde{\varepsilon}_{f,g} \subset \varepsilon_{f+g} = \{a : X \to \overline{\mathbb{R}} : a$ is e-affine and $a \leq f + g\}$.

Definition 2.14. [7] For any functions f and g, define a function $h = \sup\left\{a : a \in \tilde{\varepsilon}_{f,g}\right\}$.

The c-subdifferentiability and ε-c-subdifferentiability are developed in [6, 7] and are recalled next.

Definition 2.15. The c-subdifferential of f at $x \in X$ is a mapping $\partial_c f : X \rightrightarrows H$ defined as

$\partial_c f(x) = \{(v,w,\beta) \in H : f(y) \geq f(x) + c(y,(v,w,\beta)) - c(x,(v,w,\beta)), \quad \forall y \in X\}$,

and the ϵ-c-subdifferential of f is defined as

$\partial_{c,\epsilon} f(x) = \{(v,w,\beta) \in H : f(y) \geq f(x) + c(y,(v,w,\beta)) - c(x,(v,w,\beta)) - \epsilon, \forall y \in X\}$, for any $\epsilon \geq 0$ and all $x \in X$ such that $f(x) \in$ dom f and $\langle x,w\rangle < \beta$.

Direct consequence of the definition of $\partial_{c,\epsilon} f(x)$ yields, for all $\epsilon \geq 0$ and all $x \in$ dom f such that $\langle x,w\rangle < \beta$, we have

$$(v,w,\beta) \in \partial_{c,\epsilon} f(x) \Leftrightarrow f^c(v,w,\beta) + f(x) - c(x,(v,w,\beta)) \leq \epsilon \tag{2}$$

Remark 2.16 From the definition of c-subdifferentiability, the generalized version of Young's equality can be deduced as follows:

$$(v,w,\beta) \in \partial_c f(x) \Leftrightarrow c(x,(v,w,\beta)) = f^c(v,w,\beta) + f(x) \tag{3}$$

Next, we quote, from [7], the relation between the ε-c-subdifferential of a function f and the epigraph of its c-conjugate.

Next, we quote, from [7], the relation between the ε-c-subdifferential of a function f and the epigraph of its c-conjugate.

Proposition 2.17. Assume that f is a proper function and $x \in$ dom f, then for any $\epsilon \geq 0$,

$(v,w,\beta) \in \partial_{c,\epsilon} f(x) \Leftrightarrow (v,w,\beta,\langle x,v\rangle + \epsilon - f(x)) \in$ epi$f^{\,c}$.

In this paper, we deal with a possible infinite index set. For dealing with this situation, let us recall next the necessary definitions and results.

Definition 2.18. [12, 13] Let Ω be an index set and let $\mathcal{F}(\Omega) = \{\mathcal{M} \subseteq \Omega : \mathcal{M}$ is finite $\}$ the family of all finite subsets of Ω where $\mathcal{F}(\Omega)$ is a directed set ordered under the inclusion relation.

(i) Let $(a_i)_{i\in\Omega} \subseteq R \cup \{+\infty\}$. For every finite set $\mathcal{M} \in \mathcal{F}(\Omega)$, we associate the element $s_{\mathcal{M}} = \sum_{i\in\mathcal{M}} a_i \in R \cup \{+\infty\}$, which is called the $\mathcal{M}$-partial sum of $(a_i)_{i\in\Omega}$. The sum of $(a_i)_{i\in\Omega}$ is defined by $\sum_{i\in\Omega} a_i = \lim_{\mathcal{M}\in\mathcal{F}(\Omega)} \sum_{i\in\mathcal{M}} a_i$. Note that $\sum_{i\in\Omega} a_i \in R$ if there exists $s \in R$ such that for each $\epsilon > 0$, there exists a finite set $\mathcal{M}_0 \subset \Omega$ such that for each finite subset $\mathcal{M} \supset \mathcal{M}_0$ we have $|s_{\mathcal{M}} - s| < \epsilon$. i.e., the sum of $(a_i)_{i\in\Omega}$ is convergent. Moreover, we say that $\sum_{i\in\Omega} a_i = +\infty$ if for each $L > 0$, there exists a finite set $\mathcal{M}_0 \subset \Omega$ such that for every finite set $\mathcal{M} \supset \mathcal{M}_0$, we have that $\sum_{i\in\mathcal{M}} a_i \geq L$

(ii) Let $v \in X^*$ and let $(v_i)_{i\in\Omega}$ contained in X^*. We say that $v = \sum^*_{i\in\Omega} v_i$ if and only if $\langle v,x\rangle = \sum_{i\in\Omega} \langle v_i,x\rangle = \lim_{\mathcal{M}\in\mathcal{F}(\Omega)} \sum_{i\in\mathcal{M}} \langle v_i,x\rangle$, for each $x \in X$. The sum of $(v_i)_{i\in\Omega}$, in this case, is called weakly convergent. In other words, $\sum_{i\in\Omega} \langle v_i,x\rangle$ converges in R for each $x \in X$.

(iii) Let $\{A_i\}_{i\in\Omega} \subseteq X^*$. The sum of $\{A_i\}_{i\in\Omega}$ is defined as $\sum^*_{i\in\Omega} A_i = \{v \in X^* : \forall i \in \Omega, \exists v_i \in A_i$ such that $v = \sum^*_{i\in\Omega} v_i\}$. In general, if $\{(A_i,B_i,C_i)\}_{i\in\Omega} \subseteq H$ then $\sum^*_{i\in\Omega}(A_i,B_i,C_i) = \{(v,w,\beta) \in H : \forall i \in \Omega, \exists\, (v_i,w_i,\beta_i) \in H$ such that $(v,w,\beta) = \sum^*_{i\in\Omega}(v_i,w_i,\beta_i)\}$

Proposition 2.19 [12] Let $\{a_i,b_i,c_i\}_{i\in\Omega} \subseteq R$ be such that $a_i \leq b_i \leq c_i$ for any $i \in \Omega$. If $\sum_{i\in\Omega} a_i \in R$ and $\sum_{i\in\Omega} c_i \in R$. Then, $\sum_{i\in\Omega} b_i \in R$.

Remark 2.20. We can generalize the infinite sum described in Definition 2.18(ii) by the function c (recalled in Definition 2.5) instead of the duality pairing. Namely, let $(v,w,\beta) \in H = X^* \times X^* \times \mathbb{R}$ and let $(v_i,w_i,\beta_i)_{i\in\Omega}$ contained in H. Then, $(v,w,\beta) = \sum^*_{i\in\Omega}(v_i,w_i,\beta_i)$ if and only if $c(x,(v,w,\beta)) = \sum_{i\in\Omega} c\,(x,(v_i,w_i,\beta_i)) = \lim_{\mathcal{M}\in\mathcal{F}(\Omega)} \sum_{i\in\mathcal{M}} c\,(x,(v_i,w_i,\beta_i))$, for each $x \in X$.

Remark 2.21 In view of Definition 2.18(ii), we can extend Definition 2.12, Remark 2.13, and Definition 2.14 to infinitely many numbers of functions as follows.

Definition 2.22 Let $f_i : X \to \bar{R}$ be a function for each $i \in \Omega$ We say that $a \in \tilde{\varepsilon}_{f_i}$ is an e-affine function if there exists e-affine functions $a_i \in \varepsilon_{f_i}$ such that for each $i \in \Omega$

$$a_i(x) = \begin{cases} \langle x,v_i\rangle - \gamma_i, & \text{if } \langle x,w_i\rangle < \beta_i \\ +\infty, & \text{otherwise.} \end{cases}$$

with

$$a(x) = \begin{cases} \sum_{i\in\Omega}(\langle x,v_i\rangle - \gamma_i), & \text{if } \sum_{i\in\Omega}\langle x,w_i\rangle < \sum_{i\in\Omega}\beta_i \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus, the set

$$\tilde{\varepsilon}_{\sum_{i\in\Omega} f_i} = \left\{a : X \to \bar{R} : a \text{ is } e- \text{ affine with } a \leq \sum_{i\in\Omega} a_i \leq \sum_{i\in\Omega} f_i\right\}. \tag{4}$$

From Eq. (4), we have $\tilde{\varepsilon}_{\sum_{i\in\Omega} f_i} \subset \epsilon_{\sum_{i\in\Omega} f_i} = \{a : X \to \bar{R} : a \text{ is } e- \text{ affine with } a \leq \sum_{i\in\Omega} f_i\}$.

In this case, the function $h : X \to \bar{R}$, in Definition 2.14, is defined as $h = \sup\left\{a : a \in \tilde{\varepsilon}_{f_i}\right\}$.

3. Fenchel duality for evenly convex optimization problems

Duality theory establishes a relationship between a mathematical optimization problem, called the primal problem, and another optimization problem, called the dual problem derived from the given original (primal) problem. The attractive properties of duality have motivated researchers to investigate the notion of duality for different types of optimization problems with different structures of the objective function and the constraint set [14–18].

In this section, we consider the following primal optimization problem (P)

$$\min \sum_{i\in\Omega} f_i(x)$$
$$\text{subject to } x \in X,$$

and its Fenchel dual problem (D)

$$\max - \sum_{i\in\Omega} f_i^c\,(v_i, w_i, \beta_i)$$

$$\text{subject to } \sum_{i\in\Omega}^{*} (v_i, w_i, \beta_i) = (0,0,0) \text{ for all } i \in \Omega,$$

such that $f(x) = \sum_{i\in\Omega} f_i(x)$ and $\{f, f_i\}_{i\in\Omega}$ is a family of infinitely many e-convex and proper functions where $f, f_i : X \to R \cup \{+\infty\}$, for any $i \in \Omega$. The aim is to develop conditions which ensure

$$\inf\left\{\sum_{i\in\Omega} f_i(x) : x \in X\right\} = \max\left\{-\sum_{i\in\Omega} f_i^c\,(v_i, w_i, \beta_i) : \sum_{i\in\Omega}^{*} (v_i, w_i, \beta_i) = (0,0,0)\right\}. \tag{5}$$

The strong duality conditions we develop are based on the consideration of the epigraphs of the c-conjugate of the dual objective functions and the ε-c-subdifferential of the primal objective functions and is inspired by Lemma 3.1 and Theorem 3.2 in the work of Li and Ng [13] to obtain strong duality for lower semi continuous proper convex functions of the primal problem. To do this, we need to proof some essential properties in the framework of even convex setting. Throughout this section, and for simplicity in writing the statements, we state the following assumption.

Assumption: Let $\{f, f_i\}_{i\in\Omega}$ be a collection of e-convex and proper functions such that $f = \sum_{i\in\Omega} f_i$ where $f, f_i : X \to \mathbb{R} \cup \{+\infty\}$ for all $i \in \Omega$. We start with the following proposition which generalizes Lemma 3.1 in [13].

Proposition 3.1 Let $\{f, f_i\}_{i\in\Omega}$ are defined as in the Assumption. Then e' co $\left(\sum_{i\in\Omega}^{*} epif_i^c\right) = epif^c$

Proof Let us first show that $\sum_{i\in\Omega}^{*} epif_i^c \subseteq epif^c$. Take arbitrary $(v, w, \beta, \gamma) \in \sum_{i\in\Omega}^{*} epif_i^c$.

Then, there exists $(v_i, w_i, \beta_i, \gamma_i) \in H \times R$ such that $f_i^c\,(v_i, w_i, \beta_i) \le \gamma_i$ for each $i \in \Omega$ where $\sum_{i\in\Omega}^{*} (v_i, w_i, \beta_i) = (v, w, \beta)$ and $\sum_{i\in\Omega} \gamma_i = \gamma$. From the definition of f_i^c, we have $c\,(x, (v_i, w_i, \beta_i)) - \gamma_i \le f_i(x)$ for any $i \in \Omega$ and for any $x \in X$. Thus, from Definition 2.10, $c\,(x, (v_i, w_i, \beta_i)) - \gamma_i \in \epsilon_{f_i}$. Thus, $c\,(x, \sum_{i\in\Omega}^{*} (v_i, w_i, \beta_i)) - \sum_{i\in\Omega}^{*} \gamma_i \in \tilde{\varepsilon}_{\Sigma f_i}$. From Eq. (4) we get $c(x, (v, w, \beta)) - \gamma \le \sum_{i\in\Omega} f_i(x) = f(x)$, i.e., $c(x, (v, w, \beta)) - f(x) \le \gamma$, for all $x \in X$. This yield, $f^c(v, w, \beta) \le \gamma$ which means $(v, w, \beta, \gamma) \in$ epif ^{c} as required. Since f^c is e'-convex set.

Therefore, e' co $\left(\sum_{i\in\Omega}^{*} epif_i^c\right) \subseteq$ epif ^{c}. To show the reverse inclusion epif $^c \subseteq e'$ co $\left(\sum_{i\in\Omega}^{*}\text{ epi } {}_i^c\right)$, we employ Definition 2.10 and choose an e'-convex function $k : H \to \bar{R}$ with e' co $\left(\sum_{i\in\Omega}^{*}\text{ epifific }\right) =$ epi k. From Proposition 2.6, we have $k = k^{c'c}$ and $k^{c'}$ is e-convex function. Thus, we get

$$\sum_{i\in\Omega}^{*} epif_i^c \subset e'co\left(\sum_{i\in\Omega}^{*} epif_i^c\right) = epik = epik^{c'c}. \tag{6}$$

Next, we show that $\tilde{\varepsilon}_{\sum_{i\in\Omega} f_i} \subset \tilde{\varepsilon}_{k^{cl}}$. Take $a \in \tilde{\varepsilon}_{\sum_{i\in\Omega} f_i}$, then there exist $a_i \in \tilde{\varepsilon}_{f_i}$ such that, for any $x \in X$ and $i \in \Omega$, we get

$$a_i(x) = \begin{cases} \langle x, v_i\rangle - \gamma_i, & \text{if } \langle x, w_i\rangle < \beta_i \\ +\infty, & \text{otherwise.}\end{cases}$$

and $a_i(x) \le f_i(x)$. Then,

$$a(x) = \begin{cases} \sum_{i\in\Omega} (\langle x, v_i\rangle - \gamma_i), & \text{if } \sum_{i\in\Omega} \langle x, w_i\rangle < \sum_{i\in\Omega} \beta_i \\ +\infty, & \text{otherwise.}\end{cases}$$

such that $a(x) \le f(x)$. Therefore, $c\,(x, (v_i, w_i, \beta_i)) - \gamma_i \le f_i(x)$ for any $x \in X$ and $i \in \Omega$. Thus, from Eq. (6) and the definition of the epigraph sets we get $\sum_{i\in\Omega}^{*} (v_i, w, \beta_i, \gamma_i) \in \sum_{i\in\Omega}^{*} epif_i^c \subset epik^{c'c}$ ehich means $k^{c'c}\left(\sum_{i\in\Omega}^{*} (v_i, w, \beta_i)\right) \le \sum_{i\in\Omega} \gamma_i = \gamma$, i.e., $\sup\left\{c\left(x, \sum_{i\in\Omega}^{*} (v_i, w, \beta_i)\right) - k^{c'}(x) : x \in X\right\} \le \gamma$.

This entails $c\left(x, \sum_{i\in\Omega}^{*} (v_i, w, \beta_i)\right) - \gamma \le k^{c'}(x)$, for any $x \in X$, which means $a(x) \le k^{c'}(x)$, i.e., $a \in \tilde{\varepsilon}_k c'$. Thus, $\tilde{\varepsilon}_{\Sigma f_i} \subset \tilde{\varepsilon}_{k^{c'}}$ as required to show. The last inclusion implies $f(x) = \sum_{i\in\cap f_i}(x) \le k^{c'}(x)$ for any $x \in X$ and epi $f^c \subset epik^{c'c}$ which yields from (6), epi $f^c \subseteq e'$ co $\left(\sum_{i\in\Omega}^{*} epif_i^c\right)$

Remark 3.2 Proposition 3.1 is also valid when $\{f_i\}_{i\in\Omega}$ is finite [7] [Theorem 4]. The following result is known [13] for the case when the functions under consideration are proper closed convex functions.

Proposition 3.3 Let $\{f, f_i\}_{i\in\Omega}$ are defined as in the Assumption. Then, the following statements are equivalent.

(i) $\partial_{c,\epsilon} f(x) \subseteq \cup\left\{\sum_{i\in\Omega}^{*} \partial_{c,\epsilon_i} f_i(x) : \sum_{i\in\Omega} \epsilon_i = \epsilon, \forall \epsilon, \epsilon_i \ge 0, x \in X\right\}$

(ii) $\partial_{c,\epsilon} f(x) = \cup\left\{\sum_{i\in\Omega}^{*} \partial_{c,\epsilon_i} f_i(x) : \sum_{i\in\Omega} \epsilon_i = \epsilon, \forall \epsilon, \epsilon_i \ge 0, x \in \text{ domf }\right\}.$

(iii) $\sum_{i\in\Omega}^{*} epif_{i^c} =$ epif ^{c}.

(iv) For any $(v,w,\beta) \in H = X^* \times X^* \times R$,

$$\inf\{f(x) - c(x,(v,w,\beta)) : x \in X\} = \max \left\{ -\sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i) : \sum_{i\in\Omega}^{*} (v_i,w_i,\beta_i) = (v,w,\beta) \right\}.$$

Proof. First, we prove that [(i) $\Rightarrow$ (ii)]. Namely, let $x \in \operatorname{dom} f$ and $\varepsilon, \varepsilon_i \geq 0$ such that $\sum_{i\in\Omega} \varepsilon_i = \varepsilon$, we must show that $\sum_{i\in\Omega}^{*} \partial_{c,\varepsilon_i} f(x) \subseteq \partial_{c,\varepsilon} f_i(x)$. To this aim, let $(v,w,\beta) \in \sum_{i\in\Omega}^{*} \partial_{c,s_i} f_i(x)$, then there exists $(v_i,w_i,\beta_i) \in \partial_{c,z_i} f_i(x)$ such that $\sum_{i\in\Omega}^{*} (v_i,w_i,\beta_i) = (v,w,\beta)$

From the definitions of ε – c-subdifferential and c-conjugate of f_i

$$f_i(y) - f_i(x) \geq c(y,(v_i,w_i,\beta_i)) - c(x,(v_i,w_i,\beta_i)) - \epsilon_i$$

$$f_i^c(v_i,w_i,\beta_i) + f_i(x) - \epsilon_i \leq c(x,(v_i,w_i,\beta_i))$$

for all $y \in X$ and $i \in \Omega$. Using Eqs. (2) and (3), the last inequality yields $c(x,(v_i,w_i,\beta_i)) - \epsilon_i \leq f_i^c(v_i,w_i,\beta_i) + f_i(x) - \epsilon_i \leq c(x,(v_i,w_i,\beta_i))$

$$\begin{aligned} &\sum_{i\in\Omega} c(x,(v_i,w_i,\beta_i)) - \epsilon \leq \\ &\sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i) + f(x) - \epsilon \leq \\ &\sum_{i\in\Omega} c(x,(v_i,w_i,\beta_i)) \end{aligned} \tag{7}$$

Since $(v_i,w_i,\beta_i) \in \partial_{c,\epsilon_i} f_i(x)$ then from Proposition 2.17, for each $\epsilon_i \geq 0$ and $i \in \Omega$, we get $(v_i,w_i,\beta_i,\gamma_i) \in$ epif$_i^c$. Then $c(x,v_i,w_i,\beta_i) - f_i(x) \leq \gamma_i$, for any $x \in X$. In particular, $c(x,v_i,w_i,\beta_i) - \gamma_i \leq f_i(x) < +\infty$, for any $x \in \operatorname{dom} f$ and $i \in \Omega$ which implies $\sum_{i\in\Omega} c(x,(v_i,w_i,\beta_i)) = \sum_{i\in\Omega} \langle x, v_i \rangle \leq f(x) + \gamma < +\infty$. Thus, $\sum_{i\in\Omega} c(x,(v_i,w_i,\beta_i)) = \sum_{i\in\Omega} \langle x, v_i \rangle \in R$. From the last assertion, Proposition 2.19 and Eq. (7), we obtain $\sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i) + f(x) - \epsilon \in R$ and

$$\begin{aligned} &\sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i) + f(x) - \epsilon \leq \\ &\sum_{i\in\Omega} c(x,(v_i,w_i,\beta_i)) = \\ &\sum_{i\in\Omega} \langle x, v_i \rangle = \langle x, v \rangle = c(x,(v,w,\beta)) \end{aligned} \tag{8}$$

On the other hand, for each $z \in \operatorname{dom} f$, we have

$$\begin{aligned} &c(z,(v,w,\beta)) - f(z) = \\ &\left.\sum_{i\in\Omega} c(x,(v_i,w_i,\beta_i)) - f_i(z) \right) \leq \\ &\sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i) \end{aligned} \tag{9}$$

By combining Eqs. (8) and (9), we have $f^c(v,w,\beta) + f(x) - \epsilon \leq \sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i) + f(x) - \epsilon \leq c(x,(v,w,\beta))$

Therefore, $f^c(v,w,\beta) + f(x) - c(x,(v,w,\beta)) \leq \varepsilon$ which yields $(v,w,\beta) \in \partial_{c,z} f(x)$. Thus, $\sum_{i\in\Omega}^{*} \partial_{c,\varepsilon_i} f(x) \subseteq \partial_{c,\varepsilon} f_i(x)$ as required. Next, we show [(ii) $\Rightarrow$ (iii)]. From Proposition 3.1 and Definition 2.9, we have $\sum_{i\in\Omega}^{*} epif_i^c \subseteq e' \operatorname{co}\left(\sum_{i\in\Omega}^{*} epif_i^c\right) = epif^c$. Thus, it is enough to satisfy epi $f^c \subseteq \sum_{i\in\Omega}^{*} epif_i^c$. Let $(v,w,\beta,\gamma) \in epif^c$ then from Proposition 2.17 there exists $\varepsilon \geq 0$ such that $(v,w,\beta) \in \partial_{c,\varepsilon} f(x)$ and $\gamma =< x, v > +\varepsilon - f(x)$, for all $x \in X$. From the assumption of (ii), there exist $\varepsilon_i \geq 0$ and $(v_i,w_i,\beta_i) \in \partial_{c,\varepsilon_i} f_i(x)$. Using again Proposition 2.17, we obtain $(v_i,w_i,\beta_i,\gamma_i) \in$ epi f_i^c where $\gamma_i =< x, v_i > +\varepsilon_i - f_i(x), \sum_{i\in\Omega} \varepsilon_i = \varepsilon$, $\sum_{i\in\Omega}^{*} (v_i,w_i,\beta_i) = (v,w,\beta)$ and $\sum_{i\in\Omega} \gamma_i = \gamma$

Thus, $(v,w,\beta,\gamma) \in$ epi f^c. Now, we show [(iii) $\Rightarrow$ (iv)]. Let $(v,w,\beta) \in H$, then from Eq. (9) we have $-\sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i) \leq \inf\{c(z,(v,w,\beta)) - f(z) : z \in \operatorname{dom} f\} = -f^c(v,w,\beta)$ i.e., $\max\left\{-\sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i) : \sum_{i\in\Omega}^{*} (v_i,w_i,\beta_i) = (v,w,\beta)\right\} \leq \inf_{z\in domf}\{c(z,(v,w,\beta)) - f(z)\}$. To finish the proof, it is enough to show that there exist $(v_i,w_i,\beta_i) \in H$, for each $i \in \Omega$, such that

$$\begin{aligned} &\inf_{z\in \operatorname{dom}f} \{c(z,(v,w,\beta)) - f(z)\} \leq \\ &\max\left\{ -\sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i) : \sum_{i\in\Omega}^{*} (v_i,w_i,\beta_i) = (v,w,\beta) \right\} \end{aligned} \tag{10}$$

To do this, assume that $\inf_{z\in \operatorname{dom}f}\{c(z,(v,w,\beta)) - f(z)\} > -\infty$, i.e., $f^c(v,w,\beta) < +\infty$. Thus, $(v,w,\beta,f^c(v,w,\beta)) \in epif^c = \sum_{i\in\Omega}^{*} epif_i^c$. Then, there exist $(v_i,w_i,\beta_i,\gamma_i) \in epif_i^c$ for each $i \in \Omega$ such that

$$\begin{aligned} &\sum_{i\in\Omega}^{*} (v_i,w_i,\beta_i) = (v,w,\beta) \\ &\text{and } \sum_{i\in\Lambda} \gamma_i = f^c(v,w,\beta). \end{aligned} \tag{11}$$

Now,$(v_i,w_i,\beta_i,\gamma_i) \in$ epif$_i^c$ implies that for any $z \in X, i \in \Omega$.

$$c(z,(v_i,w_i,\beta_i)) - f_i(z) \leq f_i^c(v_i,w_i,\beta_i) \leq \gamma_i \tag{12}$$

$$\begin{aligned} &\text{i.e., } \sum_{i\in\Omega} (c(z,(v_i,w_i,\beta_i)) - f_i(z)) \leq \\ &\sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i) \leq \sum_{i\in\Omega} \gamma_i = f^c(v,w,\beta) \end{aligned} \tag{13}$$

Note that, for $z \in \operatorname{dom} f, \sum_{i\in\Omega} f_i(z) = f(z) \in R$, then from Eq. (11) and Proposition 2.19, the sum $\sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i)$ in the expression Eq. (13) belongs to R. Taking the supremum to Eq. (12) over all $z \in \operatorname{dom} f$ to get $f^c(v,w,\beta) = \sup_{z\in \operatorname{dom}f}\{c(z,(v,w,\beta)) - f(z)\} \leq \sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i) \leq \sum_{i\in\Omega} \gamma_i = f^c(v,w,\beta)$.

The last expression with (12) forces $f_i^c(v_i, w_i, \beta_i) = \gamma_i$, for all $i \in \Omega$. Thus, $\inf_{z \in \mathrm{dom} f}\{c(z,(v,w,\beta)) - f(z)\} = -f^c(v,w,\beta) = -\sum_{i\in\Omega}\gamma_i = -\sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i)$, i.e., $\inf_{z\in\mathrm{dom} f}\{c(z,(v,w,\beta)) - f(w)\} \leq \max\left\{-\sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i) : \sum^*_{i\in\Omega}(v_i,w_i,\beta_i) = (v,w,\beta)\right\}$.

Hence, we obtain Eq. (10) as required. Finally, we show [(iv) $\Rightarrow$ (i)]. Let $\epsilon \geq 0, x \in X$, and $(v,w,\beta) \in \partial_{c,\epsilon} f(x)$. From the definition of f^c, (iv) can be expressed as

$f^c((v,w,\beta) = \min\left\{\sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i) : \sum^*_{i\in\Omega}(v_i,w_i,\beta_i) = (v,w,\beta)\right\}$

Hence, there exist $(v_i,w_i,\beta_i) \in H$ such that $\sum^*_{i\in\Omega}(v_i,w_i,\beta_i) = (v,w,\beta)$ and $f^c(v,w,\beta) = \sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i)$. Thus,

$$f^c((v,w,\beta) + f(x) - c(x,(v,w,\beta)) = \sum_{i\in\Omega}\left(f_i^c(v_i,w_i,\beta_i) + f_i(x) + c(x,(v_i,w_i,\beta_i))\right). \tag{14}$$

Now, using generalized Young's inequality, for all $i \in \Omega, f_i^c(v_i,w_i,\beta_i) + f_i(x) + c(x,(v_i,w_i,\beta_i)) \geq 0$. On the other hand, since $(v,w,\beta) \in \partial_{c,\epsilon} f(x)$ then the right hand side of (14)$f^c((v,w,\beta) + f(x) - c(x,(v,w,\beta)) \leq \epsilon$. Thus, Eq. (14) implies $0 \leq \sum_{i\in\Omega}\left(f_i^c(v_i,w_i,\beta_i) + f_i(x) + c(x,(v_i,w_i,\beta_i))\right) \leq \epsilon$. It follows that, for any $i \in \Omega$, there exist $\epsilon_i \geq 0$ with $\sum_{i\in\Omega}\epsilon_i = \epsilon$ and $f_i^c(v_i,w_i,\beta_i) + f_i(x) + c(x,(v_i,w_i,\beta_i)) \leq \epsilon_i$, for each $i \in \Omega$. Thus, $(v_i,w_i,\beta_i) \in \partial_{c,\epsilon_i} f_i(x)$, for each $i \in \Omega$. Since $\sum_{i\in\Omega}{}^*(v_i,w_i,\beta_i) = (v,w,\beta)$ then $(v,w,\beta) \in \sum^*_{i\in\Omega}\partial_{c,\epsilon_i} f_i(x)$, therefore $(v,w,\beta) \in \cup\left\{\sum^*_{i\in\Omega}\partial_{c,\epsilon_i} f_i(x) : \sum_{i\in\Omega}\epsilon_i = \epsilon, \forall \epsilon, \epsilon_i \geq 0, x \in X\right\}$ which shows (i) and hence ends the proof.

Next, we obtain strong Fenchel duality stated in Eq. (5) for the primal-dual problem ((P)-(D)) as a direct consequence of Proposition 3.3.

Corollary 3.4 Let $\{f, f_i\}_{i\in\Omega}$ are defined as in the Assumption. If any of the statements (i)-(iv) in Proposition 3.3 holds then $\inf\left\{\sum_{i\in\Omega} f_i(x) : x \in X\right\} = \max\left\{-\sum_{i\in\Omega} f_i^c(v_i,w_i,\beta_i) : \sum^*_{i\in\Omega}(v_i,w_i,\beta_i) = (0,0,0)\right\}$.

Proof. The conclusion follows from any of the fourth statements stated in Proposition 3.3. For instance, assume that Proposition 3.3(iv) holds. By setting $(v,w,\beta) = (0,0,0) \in H$, then we have the conclusion.

4. Conclusion

In this paper the classical strong duality for evenly convex optimization problems is discussed. The objective functions of primal (hence, the dual) problem is considered the sum of infinitely many even convex and proper functions. The duality conditions provided in this work are based on some evenly convex concepts that generalized from their counterparts in classical convex analysis.

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