Application Investigation Of Quasi Drazin Inverse Hyponormal Operators Within The Framework Of Fuzzy Soft Set Theory In Hilbert Space

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This study aims to investigate the realm of operator theory related to Fuzzy Soft Theory. Based on fuzzy soft Hilbert spaces, a novel formula of fuzzy soft quasi-hyponormal operator via a technique of fuzzy soft Drazin inverse named the fuzzy soft n -quasi Drazin inverse hyponormal operator (FSn-QDI-hyponormal) operator. Moreover, some analytical traits are revealed for this imposed operator. In addition, the Fuzzy Soft spectrum and fuzzy soft approximate point spectrum of this class have been discussed, as well as the restriction study of class and the direct sum and tensor product are discussed.

Keywords: Hilbert space; Soft set; Fuzzy soft set; Fuzzy soft Hilbert space; Fuzzy soft Drazin invertible; Fuzzy Soft spectrum; Fuzzy Soft point spectrum; Fuzzy Soft approximate point spectrum; Hyponormal operator

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1. Introduction

Operator theory (OT) is a significant realm of mathematics, mainly functional analysis. It is an intrinsic resource that numerous engineers and scientists rely on. This theory studies operators, their merits, and behaviors on complex Hilbert spaces. Among the important categories that have received wide attention in recent decades is the category of quasihyponormal operators. With the development of research in this field, modern trends have emerged that aim to link quasi-hyponormal operators. In 1981, Gupta and Ramanujan [1] focused on analyzing the the spectral structure of class k-quasihyponormal operators and he also provided a general matrix representation for any operator belonging to this class. In 1984, Putinar [2] proposed a universal functional model for hyponormal operators, where studied that every hyponormal operator is subscalar operator. In 2016, Bachir and Altanji [3] studied a class of (p,k)-quasiposinormal operators and concluded that the generalized derivative resulting from proving Putnam-Fugede theorem is orthogonal to its kernel. In 2017, Senthilkumar and Parvatham [4] studied the spectrum the class of k-quasi*parahyponormal and proved the correspondence of non zero points to the approximate poin spectrum and joint. In 2019, Sitati [5] introduced ageneralized class G_A of A-unitary, Anormal and A-hyponormal operators and the properties of this class of operators where explained and proven A-unitary equivalence is an equivalence relation it also proves many results in terms of polar analysis of an operator T. Afterwards, In 2020, Atheab and Mohsen [6] introduced a new generalization for hyponormal operators, which gave the solvability of the λ-commuting operators and studied some properties of these operators. In 2022,Zuo and Zuo [7] introduced the class of quasi-hyponormal operators and studied some spectral properties of this class. but some of these may not be closeable. In 2022, Mohsen [8] studied some properties using (λ, μ)-commuting operator equations, such as the product of two (N, K) hyponormal operator and the product of two (h, M)-hyponormal operators, and solved the (λ, μ)-commuting operator eguations for these operators. Whereas ,In 2020, Dana and Yousefi [9] studied the generalized class of D-hyponormal operator and extended the concept of Dnormality and achieved fundamental results in proving many theorems to include a wider range of cases within Hilbert spaces. In the same year, Chellali and Benali [10] studied a new class of operators defined on complex Hilbert space,named (n, m)-power-D-hyponormal operators, which is related to the Drazin inverse through which the proposed operator is defined. They also studied some of its advantages and presented some practical examples. In 2020, Dharmarha and Ram [11] studied two generalized classes of operators linear operators on Hilbert spaces, namely the (m,n)-paranormal and (m,n)-paranormal operators. In 2022, Mesbah and Messaoudene [12] studied a special claas of D-hyponormal and D-quasihyponormal operators and investigated some properties of these operators, they also presented a generalized study of Fuglede-Putnam theorem. In the same year. Benalia [13] studied some basic properties of a new class of (n, m)-power-D-quasi-hyponormal operators generalizations closely related to the Drazin inverse. In 2023, Mohsen [14] considered the notion of an Mhyponormal operator, a novel extension of the hyponormal operator, and discussed several crucial outcomes related to it. Moreover, Shen et al. [15] introduced a class of p-quasi-nhyponormal operators and demonstrated their structural properties via the Hansen inequality and Lowner-Heinz inequality. Afterwards, in 2024, Mohsen [16] established the notion of k Quasi $(\lambda - M)$ -hyponormal Operator. In addition, AlShammari [17] formulated another generalization related to the Drazin inverse called (n, m)-Drazin normal operators. In 2024, Mohsen [18] introduced a completely new class of quasi operators, named (n, D)-quasi operators on the Hilbert spaces using Drazin inverse,also studied the scalar and power of this concept and proved of the Tensor product and direct product for this class. In 2020, Yan and Zeng [19] introduced new proposals of Cline's formula Jacobson's lemma and for Drazin inverses, generalized Drazin inverses. Whereas, In 2025, Mohsen and Barghooth [20] introduced a new class of bounded operators named the adjointable normal power operators, established several fundamental properties of this class and investigated the direct sum and tensor product for this newly proposed category .In 2025, Dana et al. [21] likewise provided some findings on the inequalities and generalized outcomes of the D-hyponormal operator. In the same year, Mohsen and Khalaf [22] introduced a new generalization of quasinormal operators, named (k, n, N)-quasi normal operators, and investigated its relationships with normal operator and k-quasinormal, n-quasinormal of this generalization according to the necessary conditions. In past years, numerous researchers, such as Campbell and Meyer [23], Duggal and Kim [24], Chen and Sheibani [25], and Qin and Lu [26] have other generalization and important developments in Drazin inverse theory and its extensions in different fields, such as Weyl's theorem and the Spectral Mapping Theorem they have also discussed several other properties that coud be generalizations that open up horizons for applications in the analysis of normal and hyponormal operators and its implementations within Hilbert spaces and Banach algebras. In this context, Zadeh [27] in 1965 introduced a significant mathematical concept called fuzzy sets. Subsequently, Molodtsov [28] in 1999 established Soft Set Theory (SS-T) as an elegant mathematical tool for treating uncertainty and assorted complex problems. In 2001, Maji et al. [29] combined the principles of fuzzy and soft into an interesting idea, which is named Fuzzy Soft Set (FSS). Fuzzy Soft Set Theory (FSs-T) is a substantial theme. Many investigators have studied numerous elegant implementations of $FSs - T$ in varied disciplines and contributed to the expansion and highlighting of considerable new characteristics on this topic. Among their pivotal contributions are, for instance [30–38]. Subsequently, In 2020, Faried et al. [39] they studied several generalized concepts on (FS) system , where presented a generalizion of the fuzzy soft orthogonal family and the fuzzy soft orthonormal family also studied fuzzy soft spectral and fuzzy soft spectral radius with other types of fuzzy soft Linear operators ,also presented a definition of fuzzy soft shift from right and left and they concluded that fuzzy soft Hilbert space is fuzzy soft self-dual. In 2021, Faried et al. [40] investigated the fuzzy soft Hilbert space principle alongside its attributes and distinct new outcomes. In 2020, Faried et al. [41] presented a fuzzy soft hermitian operator within the framework of FS-H. Furthermore, related outcomes are imposed. In 2022, Mohsen and Mousa [42] formulated an advanced class within fuzzy soft Hilbert space called fuzzy soft quasi normal operator and discussed some properties of this concept. They also investigated examples in favor of and against this operator. Several researchers have also conducted studies in semi Hilbertian space, introduced new classes and discussed some of the fundamental properties of these classes [43, 44]. In 2024, Radharamani and Nagajothi [45] studied and presented a fuzzy soft paranormal operator. They also examined the diverse key merits of this operator in FS-H. Several notions relevant to this operator are also discussed in FS-H. In 2025, Mohsen [46] proposed a new class of operators within the framework of fuzzy soft theory, named fuzzy soft k quasi-hyponormal oerator and investigated several key features also established an equivalent theorem for this class of operators in Hilbert spaces.

In this study, a new class of fuzzy soft quasi-hyponormal operators by using the Drazin inverse is introduced, namely the fuzzy soft n-quasi Drazin inverse hyponormal operator. The analytical traits of the thin new operator are investigated. the fuzzy soft spectrum and fuzzy soft approximate point spectrum of this class and the direct sum and tensor product are also examined.

2. Methods

This section introduces the key concepts in the study of FSH, which will be employed to get the main findings

Definition 2.1. [27] The set of order pair $\widetilde{G} = \{(Y, \mu_{\tilde{g}}(Y)) \mid Y \in X, \mathcal{A}(Y) \in \mathcal{J}\}$ named fuzzy set on X with a membership function $\mu_{\widetilde{G}} : X \to \Gamma$, where $\Gamma = [0, 1]$. A set $\widetilde{G}$ is also sometimes be written as $\widetilde{G} = \left\{\frac{\mu_{\widetilde{G}}(Y)}{Y} Y \in X\right\}$. The real number, $\mu_{\widetilde{G}}(Y)$,namely the membership of Y in $\widetilde{G}$.

Definition 2.2 [28] The set $\mathfrak{f}_{\mathcal{G}} = \{\mathfrak{f}(\omega) \in (X) : \omega \in \mathcal{G}\}$ named soft set over X ,with a set of parameters Σ and (X) the set of all subsets $\mathcal{X}$ such that $\mathcal{G} \subseteq \Sigma, \mathfrak{f} : \mathcal{G} \to X$, represented by $\mathfrak{f}_{\mathcal{G}}$.

Definition 2.3. [29] The soft set $\mathfrak{f}_{\mathcal{G}}$ is named fuzzy soft set (FSs) over $\mathcal{X}$, where $\mathfrak{f} : \mathcal{G} \to \Gamma^X$, has the range $\{(\omega) \in \Gamma^X : \omega \in \mathcal{G}\}$, and the class of all (FSs), indicated by FSs$(\tilde{X})$.

Definition 2.4. [40] The fuzzy soft set $(G, B) \in$ FSs$(\tilde{X})$ named a fuzzy soft point over u symbolized by $\left(\tilde{v}_{f_{G(e)}}, B\right)$, (briefly denoted by $\tilde{V}_{f_{G(e)}}$), if for the element e $\in$ B and u $\in \tilde{X}, \alpha \in (0, 1]$ is the value of the membership degree, $f_{G(e)}(v) = \begin{cases} \alpha, \text{ if } u = u_0 \in v \text{ and } e = e_0 \in B; \\ 0, \text{ if } u \in v - \{u_0\} \text{ or } e\epsilon B - \{e_0\}. \end{cases}$

Definition 2.5. [41] The set fuzzy soft vector is fuzzy soft vector space, according with the following two operations below:

i. $\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}} = \left(v^1 \widetilde{+} v^2\right)_{\left(f_{1_{G(e_1)}} + f_{1_{G(e_2)}}\right)}$ for all $\tilde{v}^1_{\cdot f_{1_{G(e1)}}}, \tilde{v}^2_{f_{\cdot 2_{G(e)}}} \in$ FS $-$ V.

ii. $\tilde{r}\tilde{v}_{f_{G(e)}} = \overline{(rv)}_{f_{G(e)}}$ for all $\tilde{v}_{f_{G(e)}} \in$ FS $-$ V and $\forall \tilde{r} \in \tilde{R}_B$.

Definition 2.6. [39] Assume $\widetilde{H}$ be fuzzy soft vector and $\mathcal{R}(\mathcal{A})$ be fuzzy soft real set. A mapping $\|\widetilde{|}\| : \widetilde{H} \to \mathcal{R}(\mathcal{A})$ namely fuzzy soft norm on $\widetilde{H}$ if $\widetilde{\| \cdot \|}$ achieves the following:

i. for all $\tilde{v}_{f_{G(e)}} \in \widetilde{H}$ $\left\|\widetilde{\tilde{v}_{f_{G(e)}}}\right\| \cong \tilde{0}$.

ii. $\left\|\tilde{v}_{f_{G(e)}}\right\| \cong \tilde{0}$ if and only if $\tilde{v}_{f_{G(e)}} \cong \tilde{0}$.

iii. for all $\tilde{v}_{f_{G(e)}} \in \widetilde{H}$, and $\tilde{r} \in \mathcal{C}(\mathcal{A})$ $\left\|\tilde{r}\tilde{v}_{f_{G(e)}}\right\| \cong |\tilde{r}| \left\|\widetilde{\tilde{v}_{f_{G(e)}}}\right\|$.

iv. for all $\tilde{v}_{f_{G(e_1)}}, \tilde{u}_{f_{G(e_2)}} \in \widetilde{H}$ obtain

$$\left\|\widetilde{\tilde{v}_{fG(e_1)} + \tilde{u}_{fG(e_2)}}\right\| \lesssim \left\|\widetilde{\tilde{v}_{f_{G(e_1)}}}\right\| + \left\|\widetilde{\tilde{u}_{f_{G(e_2)}}}\right\|$$

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The fuzzy soft vector $\widetilde{H}$ with $\widetilde{\|}\|$ indicated by $(\widetilde{H}, \widetilde{\|} \cdot \|)$, namely fuzzy soft normed space .

Definition 2.7. [39] Assume $\widetilde{H}$ be fuzzy soft vecter and $C(\mathcal{A})$ be fuzzy soft complex set. Then, the mapping $\widetilde{\langle ., . \rangle} : \widetilde{H} \times \widetilde{H} \to C(\mathcal{A})$, called fuzzy soft inner product (FS-IP) on $\widetilde{H}$ if $\widetilde{\langle ., . \rangle}$ achieves the following:

i. for all $\tilde{v}_{f_{G(e)}} \in \widetilde{H}$ yields $\left\langle \tilde{v}_{f_{G(e)}}, \widetilde{\tilde{v}_{f_{G(e)}}} \right\rangle \gtrsim \tilde{0}$.

ii. $\langle \tilde{v}_{f_{G(e)}}, \widetilde{\tilde{v}_{f_{G(e)}}} \rangle \cong \tilde{0}$ if and only if $\tilde{v}_{f_{G(e)}} \cong \tilde{0}$.

iii. for all $\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2_G(e_2)}} \in \widetilde{H}$ gains

$$\langle \tilde{v}^1_{f_{1G(e_1)}}, \widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}} \rangle \cong \overline{\left\langle \tilde{v}^2_{f_{2(e_2)}}, \widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}} \right\rangle}$$

iv. for all $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2(e_2)}} \in \widetilde{H}$ and $\tilde{r} \in \mathcal{C}(\mathcal{A})$, attains

$$\left\langle \tilde{r}\tilde{v}^1_{f_{1_{G(e_1)}}}, \widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}} \right\rangle$$

v. for all $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}$ in $\widetilde{H}$ we yield

$$\left\langle \tilde{v}^1_{f_{1_{G(e_1)}}} + \widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}}, \tilde{v}^3_{f_{3_{G(e_3)}}} \right\rangle \cong \langle \tilde{v}^1_{f_{1_{G(e_1)}}}, \widetilde{\tilde{v}^3_{f_{3_{G(e_3)}}}} \rangle + \langle \tilde{v}^2_{f_{2_{G(e_2)}}}, \widetilde{\tilde{v}^3_{f_{3_{G(e_3)}}}} \rangle.$$

The fuzzy soft vector $\widetilde{H}$ with $\widetilde{\langle ., . \rangle}$ indicated by $(\widetilde{H}, \widetilde{\langle ., . \rangle})$ namely fuzzy soft inner product space.

Definition 2.8. [41] A fuzzy soft norm space $(\widetilde{H}, \widetilde{\|}.\|)$ is called fuzzy soft complete if every fuzzy soft Cauchy sequence in it is fuzzy soft convergent sequence in it.

Theorem 2.9. [40] The fuzzy soft inner product space $(\widetilde{U}, \widetilde{\langle ., . \rangle})$ can be considered a fuzzy soft normed space with the fuzzy soft norm $\left\|\widetilde{\tilde{v}_{f_{G(e)}}}\right\| \cong \sqrt{\left\langle \tilde{v}_{f_{G(e)}}, \widetilde{\tilde{v}_f f_{G(e)}} \right\rangle}$ for all $\tilde{v}_{f_{G(e)}} \in$ FSV$(\widetilde{U})$.

Definition 2.10. [41] Assume $(\widetilde{H}, \widetilde{\langle ., . \rangle})$ be a fuzzy soft inner product space, then is fuzzy soft Hilbert space when its fuzzy soft complete, denoted by fuzzy soft Hilbert space.

Definition 2.11. [39] Let $\widetilde{H}$ be fuzzy soft Hilbert space, the operator $\widetilde{\Lambda} : \widetilde{H} \to \widetilde{H}$, named fuzzy soft linear operator (fuzzy soft linear operator), if for all $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}$ in $\widetilde{H}$ and $\tilde{\sigma}, \tilde{\rho} \in \mathcal{C}(\mathcal{A})$, then $\widetilde{\Lambda}\left(\tilde{\sigma}\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{\rho}\tilde{v}^2_{f_{f_{G(e_2)}}}\right)$.

Definition 2.12. [40] Assume $\widetilde{H}$ be fuzzy soft Hilbert space and $\widetilde{\Lambda} : \widetilde{H} \to \widetilde{H}$ be fuzzy soft operator named fuzzy soft bounded operator, if $\exists \tilde{\Sigma} \tilde{>} \tilde{0} \in R(A)$ such that $\left\|\widetilde{\Lambda}\left(\tilde{v}_{f_{G(e)}}\right)\right\| \lesssim \tilde{\Sigma} \left\|\widetilde{\tilde{v}_{f_{G(e)}}}\right\|$ for all $\tilde{v}_{f_{G(e)}} \in \widetilde{H}$.

Definition 2.13. [41] Assume $\widetilde{H}$ be fuzzy soft Hilbert space and $\widetilde{\Lambda} : \widetilde{H} \to \widetilde{H}$ be fuzzy soft bounded operator, then the fuzzy soft adjoint operator $\widetilde{\Lambda}^*$ is defined by $\left\langle \widetilde{\Lambda}\tilde{v}^1_{f_{1_{G(e_1)}}}, \widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}} \right\rangle \cong \left\langle \tilde{v}^1_{f_{1_{G(e_1)}}}, \overline{\widetilde{\Lambda}^*}\tilde{v}^2_{f_{2_{G(e_2)}}} \right\rangle$, for all $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} \in \widetilde{H}$.

Definition 2.14. [39] Assume $\widetilde{\Lambda} \in \widetilde{B}(\widetilde{H})$ where $\widetilde{H}$ is fuzzy soft Hilbert space Then $\tilde{\sigma}(\widetilde{\Lambda})$ named fuzzy soft spectrum of fuzzy soft linear operator $\widetilde{\Lambda}$ if $\tilde{\sigma}(\widetilde{\Lambda}) \cong \{\tilde{\lambda}\tilde{\epsilon}\mathbb{C}(A) : |\tilde{\lambda}| \tilde{\leq} \|\tilde{\Lambda}\|\}$.

Definition 2.15. [39] Assume $\widetilde{\Lambda} \in \widetilde{B}(\widetilde{H})$ named fuzzy soft point spectrum of fuzzy soft linear operator $\widetilde{\Lambda}$ if there exists a non-zero fuzzy soft element $\tilde{v}_{f_{G(e)}} \tilde{\in} \widetilde{H}$ such that $(\widetilde{\Lambda} - \tilde{\lambda})\tilde{v}_{f_{G(e)}} \cong \tilde{0}$, symbolized by $\tilde{\sigma}_p(\widetilde{\Lambda})$, such that $\tilde{\sigma}_p(\widetilde{\Lambda}) \tilde{\subset} \tilde{\sigma}(\widetilde{\Lambda})$..

Definition 2.16. [39] Assume $\tilde{\lambda}\tilde{\in}C(A)$ named fuzzy soft joint point spectrum of fuzzy soft linear operator $\widetilde{\Lambda}$ if there exists a non-zero fuzzy soft element $\tilde{v}_{f_{G(e)}} \tilde{\in} \widetilde{H}$ such that $(\widetilde{\Lambda}^* - \overline{\tilde{\lambda}})\tilde{v}_{f_{G(e)}} \cong \tilde{0}$, symbolized by $\tilde{\sigma}_{jp}(\widetilde{\Lambda})$.

Definition 2.17. [39] Assume $\tilde{\lambda}\tilde{\epsilon}C(A)$ named fuzzy soft approximate point spectrum of FSlinear operator $\widetilde{\Lambda}$ if there exists a fuzzy soft sequence of fuzzy soft elements $\tilde{v}^n_{f_{n_{G(e_n)}}}$ in $\widetilde{H}$, where $\left\| \tilde{v}^n_{f_{n_{G(e_n)}}} \right\| \cong 1$ and $\widetilde{\Lambda}\tilde{v}^n_{f_{n_{G(e_n)}}} - \tilde{\lambda}\tilde{v}^n_{f_{n_{G(e_n)}}} \to \tilde{0}$, symbolized by $\tilde{\sigma}_a(\widetilde{\Lambda})$, such that $\tilde{\sigma}_a(\widetilde{\Lambda}) \tilde{\subset} \tilde{\sigma}(\widetilde{\Lambda})$.

Definition 2.18. [39] Assume $\tilde{\lambda}\tilde{\epsilon}C(A)$ named fuzzy soft continuous spectrum of fuzzy soft linear operator $\widetilde{\Lambda}$ if $(\tilde{\lambda}\tilde{I} - \widetilde{\Lambda})$ is fuzzy soft injective and has fuzzy soft dense range in $\widetilde{H}$, is fuzzy soft singular and has no fuzzy soft inverse, symbolized by $\tilde{\sigma}_c(\widetilde{\Lambda})$, such that $\tilde{\sigma}_p(\widetilde{\Lambda}) \cup \tilde{\sigma}_c(\widetilde{\Lambda}) \tilde{\subset} \tilde{\sigma}_a(\widetilde{\Lambda})$.

3. Results and discussion

Definition 3.1 Assume $\widetilde{\Lambda} \in \widetilde{B}(\widetilde{H})$ named fuzzy soft n - quasi Drazin inverse hyponormal operator if $\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D \geq \widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^*$, and shortly (FSn-QDI-hyponormal) opertator.

Example 3.2. $\widetilde{\Lambda} \cong \begin{bmatrix} (0.9,0) & (0.3,0) \\ (0.1,0) & (0.4,0) \end{bmatrix}$ is FSn-QDI-hyponormal opertator, since

$\widetilde{\Lambda}^* \cong \begin{bmatrix} (0.9,0) & (-0.3,0) \\ (-0.1,0) & (0.4,0) \end{bmatrix}$ and for

n = 2 $\in \mathbb{N}\widetilde{\Lambda}^2 \cong \begin{bmatrix} (0.84,0) & (0.39,0) \\ (0.13,0) & (0.19,0) \end{bmatrix}$ and

$\widetilde{\Lambda}^D \cong \begin{bmatrix} (1,0) & (0.6,0) \\ (0.2,0) & (0,0) \end{bmatrix}$ then $\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D \geq \widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^*$.

Proposition 3.3. Assume $\widetilde{\Lambda} : \widetilde{H} \to \widetilde{H}$ be a FSn-QDI-hyponormal operator on fuzzy soft Hilbert space $\widetilde{H}$, then $\left(\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D\right)^n \geq \left(\widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^*\right)^n$ FSn-QDI-Hyponormal operator.

Proof: By utilizing mathematical induction, since $\widetilde{\Lambda}$ is an FSn-QDI-hyponormal operator, then $\left(\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D\right)^n \geq \left(\widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^*\right)^n$ is FSn-QDI-hyponormal operator.

For n = 1 then $\left(\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D\right)^1 \tilde{\geq} \left(\widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^*\right)^1 \ldots$ (i)

Given this outcome is attained for n = k, consequently

$\left(\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D\right)^k \cong \left(\widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^*\right)^k \ldots$ (ii)

Then, to demonstrate the validity of the outcome for n = k + 1,

$\left(\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D\right)^{k+1} \cong \left(\widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^*\right)^k \left(\widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^*\right)^1$,

from Equality (i) and Equality (ii), we yield

$\left(\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D\right)^{k+1} \tilde{\geq} \left(\widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^*\right)^{k+1}$

Therefore $\left(\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D\right)^n \cong \left(\widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^*\right)^n$ is FSn-QDI-hyponormal operator.

Theorem 3.4. Assume $\widetilde{\Lambda}$ and $\widetilde{W}$ FSn-QDI-hyponormal operator on fuzzy soft Hilbert space $\widetilde{H}$. If $\widetilde{W}\widetilde{\Lambda} \cong \widetilde{\Lambda}\widetilde{W}, \widetilde{\Lambda}^* \widetilde{W} \cong \widetilde{W}\widetilde{\Lambda}^*$ then $\widetilde{\Lambda} + \widetilde{W}$ is FSn-QDI-hyponormal operator

Proof: Since $\widetilde{\Lambda}$ and $\widetilde{W}$ are FSn-QDI-hyponormal operator with $\widetilde{W}\widetilde{\Lambda} \cong \widetilde{\Lambda}\widetilde{W}, \widetilde{\Lambda}^* \widetilde{W} \cong \widetilde{W}\widetilde{\Lambda}^*$.

$$
\begin{aligned}
&(\tilde{\Lambda} + \tilde{W})^n(\tilde{\Lambda} + \tilde{W})^*(\tilde{\Lambda} + \tilde{W})^D \\
&\cong (\tilde{\Lambda}^n + \tilde{W}^n)(\tilde{\Lambda}^* + \tilde{W}^*)\left(\tilde{\Lambda}^D + \tilde{W}^D\right) \\
&\cong \left(\tilde{\Lambda}^n\tilde{\Lambda}^* + \tilde{W}^n\tilde{W}^*\right)\left(\tilde{\Lambda}^D + \tilde{W}^D\right) \\
&\cong \tilde{\Lambda}^n\tilde{\Lambda}^*\tilde{\Lambda}^D + \tilde{W}^n\tilde{W}^*\tilde{W}^D \\
&\cong \tilde{\Lambda}^D\tilde{\Lambda}^n\tilde{\Lambda}^* + \tilde{W}^D\tilde{W}^n\tilde{W}^* \\
&\cong \left(\tilde{\Lambda}^D + \tilde{W}^D\right)\left(\tilde{\Lambda}^n\tilde{\Lambda}^* + \tilde{W}^n\tilde{W}^*\right) \\
&\cong \left(\tilde{\Lambda}D + \tilde{W}^D\right)(\tilde{\Lambda}^n + \tilde{W}^n)(\tilde{\Lambda}^* + \tilde{W}^*) \\
&\cong (\tilde{\Lambda} + \tilde{W})^D(\tilde{\Lambda} + \tilde{W})^n(\tilde{\Lambda} + \tilde{W})^*
\end{aligned}
$$

Hence, $\widetilde{\Lambda} + \widetilde{W}$ is FSn-QDI-hyponormal operator.

Theorem 3.5. Assume $\widetilde{\Lambda}$ and $\widetilde{W}$ be FSn-QDI-hyponormal operator on fuzzy soft Hilbert space $\widetilde{H}$ If $\widetilde{\Lambda}^n\widetilde{W}^D \cong \widetilde{W}^D\widetilde{\Lambda}^n, \widetilde{\Lambda}^D\widetilde{W}^* \cong \widetilde{W}^*\widetilde{\Lambda}^D, \widetilde{\Lambda}^*\widetilde{W}^n \cong \widetilde{W}^n\widetilde{\Lambda}^*, \widetilde{\Lambda}^*\widetilde{W}^D \cong \widetilde{W}^D\widetilde{\Lambda}^*$, then $\widetilde{\Lambda}\widetilde{W}$ is FSn-QDIhyponormal operator.

Proof: Since $\widetilde{\Lambda}$ and $\widetilde{W}$ are FSn-QDI-hyponormal operator with $\widetilde{\Lambda}^n\widetilde{W}^D \cong \widetilde{W}^D\widetilde{\Lambda}^n$, $\widetilde{\Lambda}^D\widetilde{W}^* \cong \widetilde{W}^*\widetilde{\Lambda}^D$, $\widetilde{\Lambda}^*\widetilde{W}^n \cong \widetilde{W}^n\widetilde{\Lambda}^*, \widetilde{\Lambda}^*\widetilde{W}^D \cong \widetilde{W}^D\widetilde{\Lambda}^*$.

$$
\begin{aligned}
&(\widetilde{\Lambda}\widetilde{W})^n(\widetilde{\Lambda}\widetilde{W})^*(\widetilde{\Lambda}\widetilde{W})^D \\
&\cong \widetilde{\Lambda}^n\widetilde{W}^n\widetilde{\Lambda}^*\widetilde{W}^*\widetilde{\Lambda}^D\widetilde{W}^D \cong \widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{W}^n\widetilde{W}^*\widetilde{\Lambda}^D\widetilde{W}^D \\
&\cong \widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{W}^n\widetilde{\Lambda}^D\widetilde{W}^*\widetilde{W}^D \cong \widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D\widetilde{W}^n\widetilde{W}^*\widetilde{W}^D \\
&\tilde{\geq} \widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{W}^D\widetilde{W}^n\widetilde{W}^* \cong \widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{W}^D\widetilde{\Lambda}^*\widetilde{W}^n\widetilde{W}^* \\
&\cong \widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{W}^D\widetilde{W}^n\widetilde{\Lambda}^*\widetilde{W}^* \cong \widetilde{\Lambda}^D\widetilde{W}^D\widetilde{\Lambda}^n\widetilde{W}^n\widetilde{\Lambda}^*\widetilde{W}^* \\
&\cong (\widetilde{\Lambda}\widetilde{W})^D(\widetilde{\Lambda}\widetilde{W})^n(\widetilde{\Lambda}\widetilde{W})^*
\end{aligned}
$$

Hence, $\widetilde{\Lambda}\widetilde{W}$ is FSn-QDI-hyponormal operator.

Definition 3.6. The fuzzy soft tensor product of $\widetilde{\Lambda}_1, \widetilde{\Lambda}_2, \ldots\ldots, \widetilde{\Lambda}_m$ is defined as:

$$\left(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \ldots\ldots \otimes \widetilde{\Lambda}_m\right)(e_1 \ldots e_n) =$$

$$\widetilde{\Lambda}_1(e_1 \ldots e_n) \otimes \widetilde{\Lambda}_2(e_1 \ldots\ldots e_n) \otimes \ldots\ldots \otimes \widetilde{\Lambda}_m(e_1 \ldots e_n)$$

Theorem 3.7. Assume $\widetilde{\Lambda}_1, \widetilde{\Lambda}_2, \widetilde{\Lambda}_3, \ldots\ldots, \widetilde{\Lambda}_m$ are FSn-QDI-hyponormal operator in $\widetilde{B}(\widetilde{H}) \forall k = 1,2,3,\ldots\ldots, m$ then $\left(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \ldots\ldots \otimes \widetilde{\Lambda}_m\right)$ is FSn-QDI-hyponormal operator.

Proof:

$$(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \ldots \otimes \widetilde{\Lambda}_m)^n \left(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \ldots \otimes \widetilde{\Lambda}_m\right)^* \left(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \ldots \otimes \widetilde{\Lambda}_m\right)^D$$
$$\cong \left(\widetilde{\Lambda}_1^n \otimes \widetilde{\Lambda}_2^n \otimes \ldots \otimes \widetilde{\Lambda}_m^n\right)\left(\widetilde{\Lambda}_1^* \otimes \widetilde{\Lambda}_2^* \otimes \ldots \otimes \widetilde{\Lambda}_m^*\right)\left(\widetilde{\Lambda}_1^D \otimes \widetilde{\Lambda}_2^D \otimes \ldots \otimes \widetilde{\Lambda}_m^D\right)$$
$$\cong \left(\widetilde{\Lambda}_1^n \otimes \widetilde{\Lambda}_2^n \otimes \ldots. \otimes \widetilde{\Lambda}_m^n\right)\left(\widetilde{\Lambda}_1^*\widetilde{\Lambda}_1^D \otimes \widetilde{\Lambda}_2^*\widetilde{\Lambda}_2^D \otimes \ldots \otimes \widetilde{\Lambda}_m^*\widetilde{\Lambda}_m^D\right)$$
$$\cong \left(\widetilde{\Lambda}_1^n\widetilde{\Lambda}_1^*\widetilde{\Lambda}_1^D \otimes \widetilde{\Lambda}_2^n\widetilde{\Lambda}_2^*\widetilde{\Lambda}_2^D \otimes \ldots. \otimes \widetilde{\Lambda}_m^n\widetilde{\Lambda}_m^*\widetilde{\Lambda}_m^D\right)$$
$$\widetilde{\geq} \left(\widetilde{\Lambda}_1^D\widetilde{\Lambda}_1^n\widetilde{\Lambda}_1^* \otimes \widetilde{\Lambda}_2^D\widetilde{\Lambda}_2^n\widetilde{\Lambda}_2^* \otimes \ldots \otimes \widetilde{\Lambda}_m^D\widetilde{\Lambda}_m^n\widetilde{\Lambda}_m^*\right)$$
$$\cong \left(\widetilde{\Lambda}_1^D \otimes \widetilde{\Lambda}_2^D \otimes \ldots \otimes \widetilde{\Lambda}_m^D\right)\left(\widetilde{\Lambda}_1^n\widetilde{\Lambda}_1^* \otimes \widetilde{\Lambda}_2^n\widetilde{\Lambda}_2^* \otimes \ldots \otimes \widetilde{\Lambda}_m^n\widetilde{\Lambda}_m^*\right)$$
$$\cong \left(\widetilde{\Lambda}_1^D \otimes \widetilde{\Lambda}_2^D \otimes \ldots \otimes \widetilde{\Lambda}_m^D\right)\left(\widetilde{\Lambda}_1^n \otimes \widetilde{\Lambda}_2^n \otimes \ldots \otimes \widetilde{\Lambda}_m^n\right)\left(\widetilde{\Lambda}_1^* \otimes \widetilde{\Lambda}_2^* \otimes \ldots \otimes \widetilde{\Lambda}_m^*\right)$$
$$\cong \left(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \ldots \otimes \widetilde{\Lambda}_m\right)^D \left(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \ldots \otimes \widetilde{\Lambda}_m\right)^n (\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \ldots \otimes \widetilde{\Lambda}_m)^*$$

Hence, $\left(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \ldots \otimes \widetilde{\Lambda}_m\right)$ is FSn-QDI-hyponormal operator.

Definition 3.8. The fuzzy soft tensor sum of $\widetilde{\Lambda}_1, \widetilde{\Lambda}_2, \ldots, \widetilde{\Lambda}_m$ is defined as: $\left(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \ldots \oplus \widetilde{\Lambda}_m\right)(e_1 \ldots e_n)$
$= \widetilde{\Lambda}_1 (e_1 \ldots e_n) \oplus \widetilde{\Lambda}_2 (e_1 \ldots e_n) \oplus \ldots \oplus \widetilde{\Lambda}_m (e_1 \ldots e_n)$.

Theorem 3.9. Assume $\widetilde{\Lambda}_1, \widetilde{\Lambda}_2, \widetilde{\Lambda}_3, \ldots, \widetilde{\Lambda}_m$ are FSn-QDI-hyponormal operator in $\widetilde{B}(\widetilde{H})$ for all $k = 1,2,3,\ldots, m$, then $\left(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \ldots \bigoplus \widetilde{\Lambda}_m\right)$ is FSn-QDI-hyponormal operator.

Proof:

$$\left(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \ldots \oplus \widetilde{\Lambda}_m\right)^n \left(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \ldots \oplus \widetilde{\Lambda}_m\right)^* \left(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \ldots \oplus \widetilde{\Lambda}_m\right)^D$$
$$\cong \left(\widetilde{\Lambda}_1^n \oplus \widetilde{\Lambda}_2^n \oplus \ldots \oplus \widetilde{\Lambda}_m^n\right)\left(\widetilde{\Lambda}_1^* \oplus \widetilde{\Lambda}_2^* \oplus \ldots \oplus \widetilde{\Lambda}_m^*\right)\left(\widetilde{\Lambda}_1^D \oplus \widetilde{\Lambda}_2^D \oplus \ldots \oplus \widetilde{\Lambda}_m^D\right)$$
$$\cong \left(\widetilde{\Lambda}_1^n \oplus \widetilde{\Lambda}_2^n \oplus \ldots. \oplus \widetilde{\Lambda}_m^n\right)\left(\widetilde{\Lambda}_1^*\widetilde{\Lambda}_1^D \oplus \widetilde{\Lambda}_2^*\widetilde{\Lambda}_2^D \oplus \ldots. \widetilde{\Lambda}_m^*\widetilde{\Lambda}_m^D\right)$$
$$\cong \left(\widetilde{\Lambda}_1^n\widetilde{\Lambda}_1^*\widetilde{\Lambda}_1^D \oplus \widetilde{\Lambda}_2^n\widetilde{\Lambda}_2^*\widetilde{\Lambda}_2^D \oplus \ldots \oplus \widetilde{\Lambda}_m^n\widetilde{\Lambda}_m^*\widetilde{\Lambda}_m^D\right)$$
$$\widetilde{\geq} \left(\widetilde{\Lambda}_1^D\widetilde{\Lambda}_1^n\widetilde{\Lambda}_1^* \oplus \widetilde{\Lambda}_2^D\widetilde{\Lambda}_2^n\widetilde{\Lambda}_2^* \oplus \ldots. \oplus \widetilde{\Lambda}_m^D\widetilde{\Lambda}_m^n\widetilde{\Lambda}_m^*\right) \text{ since by definition}$$
$$\text{(FSn-QDI-hyponormal) } \widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D \geq \widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^*$$
$$\cong \left(\widetilde{\Lambda}_1^D \oplus \widetilde{\Lambda}_2^D \oplus \ldots \oplus \widetilde{\Lambda}_m^D\right)\left(\widetilde{\Lambda}_1^n\widetilde{\Lambda}_1^* \oplus \widetilde{\Lambda}_2^n\widetilde{\Lambda}_2^* \oplus \ldots \oplus \widetilde{\Lambda}_m^n\widetilde{\Lambda}_m^*\right)$$
$$\cong \left(\widetilde{\Lambda}_1^D \oplus \widetilde{\Lambda}_2^D \oplus \ldots \oplus \widetilde{\Lambda}_m^D\right)\left(\widetilde{\Lambda}_1^n \oplus \widetilde{\Lambda}_2^n \oplus \ldots \oplus \widetilde{\Lambda}_m^n\right)\left(\widetilde{\Lambda}_1^* \oplus \widetilde{\Lambda}_2^* \oplus \ldots \oplus \widetilde{\Lambda}_m^*\right)$$
$$\cong \left(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \ldots \oplus \widetilde{\Lambda}_m\right)^D \left(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \ldots \oplus \widetilde{\Lambda}_m\right)^n \left(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \ldots \oplus \widetilde{\Lambda}_m\right)^*$$

Hence, $\left(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \ldots \bigoplus \widetilde{\Lambda}_m\right)$ is FSn-QDI-hyponormal operator.

Proposition 3.10. Assume $\widetilde{\Lambda}, \widetilde{W} \in \widetilde{B}(\widetilde{H})$, if $\widetilde{W}$ is FSn-QDI-hyponormal operator, and $\widetilde{\Lambda}$ is fuzzy soft unitary equivalent to $\widetilde{W}$ then $\widetilde{\Lambda}$ FSn-QDI-hyponormal operator.

Proof: Assume $\widetilde{W}$ is FSn-QDI-hyponormal operator and $\widetilde{\Lambda} \in \widetilde{B}(\widetilde{H})$ is FS-unitary equivalent to $\widetilde{W}$. there exists a fuzzy soft unitary equivalent $\widetilde{U} \widetilde{\in} \widetilde{B}(\widetilde{H})$, let $\widetilde{\Lambda}^n \cong \widetilde{U}^* \widetilde{W}^n \widetilde{U}$. So $\widetilde{\Lambda}^* \cong \widetilde{U}^* \widetilde{W}^* \widetilde{U}$ and $\widetilde{\Lambda}^D \cong \widetilde{U}^* \widetilde{W}^D \widetilde{U}$. We obtain

$$\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D \cong \widetilde{U}^* \widetilde{W}^n \widetilde{U} U^* \widetilde{W}^* \widetilde{U}\widetilde{U}^* \widetilde{W}^D \widetilde{U}$$

since $\widetilde{\Lambda}^n \cong \widetilde{U}^* \widetilde{W}^n \widetilde{U}$, $\widetilde{\Lambda}^* \cong \widetilde{U}^* \widetilde{W}^* \widetilde{U}$ and $\widetilde{\Lambda}^D \cong \widetilde{U}^* \widetilde{W}^D \widetilde{U}$

$$\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D \cong \widetilde{U}^* \widetilde{W}^n \widetilde{U} U^* \widetilde{W}^* \widetilde{U}\widetilde{U}^* \widetilde{W}^D \widetilde{U}$$
$$\cong \widetilde{U}^* \widetilde{W}^n \widetilde{U}\widetilde{U}^* \widetilde{W}^* \widetilde{W}^D \widetilde{U}$$
$$\cong \widetilde{U}^* \widetilde{W}^n \widetilde{W}^* \widetilde{W}^D \widetilde{U}$$
$$\widetilde{\geq} \widetilde{U}^* \widetilde{W}^D \widetilde{W}^n \widetilde{W}^* \widetilde{U} \quad \text{since by definition}$$

(FSn-QDI-hyponormal)

$$\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D \geq \widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^*$$
$$\cong \widetilde{U}^* \widetilde{W}^D \widetilde{U}\widetilde{U}^* \widetilde{W}^n \widetilde{W}^* \widetilde{U}$$
$$\cong \widetilde{U}^* \widetilde{W}^D \widetilde{U}\widetilde{U}^* \widetilde{W}^n \widetilde{U}\widetilde{V}^* \widetilde{W}^* \widetilde{U}$$
$$\cong \widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^*$$

Hence, $\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D - \widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^* \geq \widetilde{0}$.

Remark 3.11. Let $\widetilde{\Lambda} \in \widetilde{B}(\widetilde{H})$. Then:

1. $\widetilde{\Lambda}^\pi \cong I - \widetilde{\Lambda}\widetilde{\Lambda}^D$ is the spectral idempotent of $\widetilde{\Lambda}$ that corresponds to $\{\widetilde{0}\}$.

2. $\widetilde{\Lambda} \cong \widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2$, where $\widetilde{\Lambda}_1$ is invertible and $\widetilde{\Lambda}_2$ be nilpotent, is a matrix form of A according to the decomposition $\widetilde{H} \cong \overline{R\left(\widetilde{\Lambda}^\pi\right)} \oplus \text{Ker}\left(\widetilde{\Lambda}^\pi\right) \overline{R\left(\widetilde{\Lambda}^\pi\right)}$ is the closure of $R\left(\widetilde{\Lambda}^\pi\right)$.

Theorem 3.12. If $\widetilde{\Lambda} \in$ FSn − QDI-hyponormal operator, then $\widetilde{\Lambda} \cong \begin{pmatrix} \widetilde{\Lambda}_1 & \widetilde{\Lambda}_2 \\ \widetilde{0} & \widetilde{\Lambda}_3 \end{pmatrix}$ on $\widetilde{H} \cong \overline{\text{ran}\left(\widetilde{\Lambda}^D\right)} \oplus \text{Ker}\left(\widetilde{\Lambda}^D\right)$ and $\widetilde{\Lambda}_3^k \cong \widetilde{0}$ where k $\cong$ ind$(\widetilde{\Lambda})$.

Proof: Assume $\widetilde{P}$ be the orthogonal projection on $\overline{\text{ran}\left(\widetilde{\Lambda}^D\right)}$. Then $\widetilde{\Lambda}_1 \cong \widetilde{\Lambda}\widetilde{P} \cong \widetilde{P}\widetilde{\Lambda}$.

Since $\widetilde{\Lambda}$ is FSn-QDI-hyponormal operator, we have $\widetilde{P}\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D\widetilde{P} \geq \widetilde{P}\widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{P}$ implies $\widetilde{\Lambda}_1^n\widetilde{\Lambda}_1^*\widetilde{\Lambda}_1^D \geq \widetilde{\Lambda}_1^D\widetilde{\Lambda}_1^n\widetilde{\Lambda}_1^*$. Hence $\widetilde{\Lambda}_1 \in$ FSn-QDIhyponormal operator

By remark 3.11, for any $\tilde{v}_{f_{G(e)}} \cong (\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}) \in \widetilde{H}$. Then

$$\left\langle \widetilde{\Lambda}_3^k \tilde{v}^2_{f_{G(e_2)}}, \tilde{v}^2_{f_{G(e_2)}} \right\rangle \cong \left\langle \widetilde{\Lambda}^k (\mathrm{I} - \widetilde{P})\tilde{v}_{f_{G(e)}}, (\mathrm{I} - \widetilde{P})\tilde{v}_{f_{G(e)}} \right\rangle$$
$$\cong \left\langle (\mathrm{I} - \widetilde{P})\tilde{v}_{f_{G(e)}}, \widetilde{\Lambda}^{*K} (\mathrm{I} - \widetilde{P})\tilde{v}_{f_{G(e)}} \right\rangle$$
$$\cong \widetilde{0}$$

This implies $\widetilde{\Lambda}_3^k \cong \widetilde{0}$.

Lemma 3.13. If $\widetilde{\Lambda} \in$ FSn-QDI-hyponormal operator, then the restriction $\widetilde{\Lambda}_{|\widetilde{M}}$ of a FSn-QDIhyponormal operator $\widetilde{\Lambda}$ to closed subspace $\widetilde{M}$ of $\widetilde{H}$ is FSn-QDI-hyponormal operator.

Proof: Assume $\widetilde{\Lambda} \cong \begin{pmatrix} \widetilde{\Lambda}_1 & \tilde{X} \\ \widetilde{0} & \widetilde{\Lambda}_3 \end{pmatrix}$ on $\widetilde{H} \cong \widetilde{M} \bigoplus \widetilde{M}^{\widetilde{I}}$,Since $\widetilde{\Lambda} \in$ FSn-QDI-hyponormal operator, we have: $\widetilde{\Lambda}^n\widetilde{\Lambda}^*\widetilde{\Lambda}^D \geq \widetilde{\Lambda}^D\widetilde{\Lambda}^n\widetilde{\Lambda}^*$.

Hence:

$$\begin{pmatrix} \widetilde{\Lambda}_1^n & \tilde{0} \\ \widetilde{0} & \widetilde{\Lambda}_3^n \end{pmatrix} \begin{pmatrix} \widetilde{\Lambda}_1^* & \tilde{0} \\ \widetilde{0} & \widetilde{\Lambda}_2^* \end{pmatrix} \begin{pmatrix} \widetilde{\Lambda}_1^D & \tilde{X} \\ \widetilde{0} & \widetilde{\Lambda}_2^D \end{pmatrix} \gtrsim \begin{pmatrix} \widetilde{\Lambda}_1^D & \tilde{X} \\ \widetilde{0} & \widetilde{\Lambda}_2^D \end{pmatrix} \begin{pmatrix} \widetilde{\Lambda}_1^n & \tilde{X} \\ \widetilde{0} & \widetilde{\Lambda}_2^n \end{pmatrix} \begin{pmatrix} \widetilde{\Lambda}_1^* & \tilde{X} \\ \widetilde{0} & \widetilde{\Lambda}_2^* \end{pmatrix}$$

$$\begin{pmatrix} \widetilde{\Lambda}_1^n & \tilde{X} \\ \widetilde{0} & \widetilde{\Lambda}_3^n \end{pmatrix} \begin{pmatrix} \widetilde{\Lambda}_1^* & \tilde{X} \\ \widetilde{0} & \widetilde{\Lambda}_2^* \end{pmatrix} \begin{pmatrix} \widetilde{\Lambda}_1^D & \tilde{X} \\ \widetilde{0} & \widetilde{\Lambda}_2^D \end{pmatrix} - \begin{pmatrix} \widetilde{\Lambda}_1^D & \tilde{X} \\ \widetilde{0} & \widetilde{\Lambda}_2^D \end{pmatrix} \begin{pmatrix} \widetilde{\Lambda}_1^n & \tilde{X} \\ \widetilde{0} & \widetilde{\Lambda}_2^n \end{pmatrix} \begin{pmatrix} \widetilde{\Lambda}_1^* & \tilde{X} \\ \widetilde{0} & \widetilde{\Lambda}_2^* \end{pmatrix} \tilde{\geq} \tilde{0}$$

Therefore,

$$\begin{pmatrix} \widetilde{\Lambda}_1^n \widetilde{\Lambda}_1^* \widetilde{\Lambda}_1^D - \widetilde{\Lambda}_1^D \widetilde{\Lambda}_1^n \widetilde{\Lambda}_1^* & \tilde{B} \\ \tilde{F} & \tilde{G} \end{pmatrix} \geq \widetilde{0}$$

For some operators $\tilde{B}, \tilde{F}$ and $\tilde{G}$. Therefore, $\widetilde{\Lambda}_1^n \widetilde{\Lambda}_1^* \widetilde{\Lambda}_1^D - \widetilde{\Lambda}_1^D \widetilde{\Lambda}_1^n \widetilde{\Lambda}_1^* \widetilde{0}$

Hence,

$$\widetilde{\Lambda}_1^n \widetilde{\Lambda}_1^* \widetilde{\Lambda}_1^D \cong \widetilde{\Lambda}_1^D \widetilde{\Lambda}_1^n \widetilde{\Lambda}_1^*$$

This implies that $\widetilde{\Lambda}_1 \cong \widetilde{\Lambda}_{|\widetilde{M}}$ is FSn-QDI-hyponormal operator.

Remark 3.14. $\begin{pmatrix} \tilde{X} & \tilde{Y} \\ \tilde{Y}^* & \tilde{G} \end{pmatrix} \geq \tilde{0}$ if and only if $\tilde{X}, \tilde{G} \geq \tilde{0}$ and $\tilde{Y} \cong \tilde{X}^{\frac{1}{2}} \widetilde{W}^{\frac{1}{2}}$ for some contractions $\widetilde{W}$.

Lemma 3.15. Assume $\widetilde{\Lambda}$ be FSn-QDI-hyponormal operator on fuzzy soft Hilbert space $\widetilde{H}$ and $\tilde{\lambda} \not\cong \tilde{0}$ then $\widetilde{\Lambda} \tilde{v}_{f_{G(e)}} \cong \tilde{\lambda} \tilde{v}_{f_{G(e)}}$ implies $\widetilde{\Lambda}^* \tilde{v}_{f_{G(e)}} \cong \overline{\tilde{\lambda}} \tilde{v}_{f_{G(e)}}$.

Proof: assume $\tilde{v}_{f_{G(e)}} \not\cong \widetilde{0}$ and assume $\widetilde{M}_0$ be a span of $\widetilde{0}$ then is an invariant subspace of $\widetilde{\Lambda}$ and $\widetilde{\Lambda} \cong \begin{pmatrix} \tilde{\mu} & \widetilde{\Lambda}_2 \\ \tilde{0} & \widetilde{\Lambda}_3 \end{pmatrix}$, on $\widetilde{H} \cong \widetilde{M}_0 \bigoplus \widetilde{M}_0{}^{\tilde{I}}$ Assume $\widetilde{P}$ be orthogonal projection onto $\widetilde{M}_0$ and $\widetilde{\Lambda}_2 \cong \widetilde{0}$.

Since $\widetilde{\Lambda}$ is FSn-QDI-hyponormal operator, we have $\widetilde{P} \widetilde{\Lambda}^n \widetilde{\Lambda}^* \widetilde{\Lambda}^D \widetilde{P} \geq \widetilde{P} \widetilde{\Lambda}^D \widetilde{\Lambda}^n \widetilde{\Lambda}^* \widetilde{P}$

Bu simple calculations, we have $\sum \widetilde{\Lambda}_2^n \widetilde{\Lambda}_3^* \cong \widetilde{0}$

Since $\widetilde{\Lambda}$ is FSn-QDI-hyponormal operator

$$\widetilde{\Lambda}^{*k} \left(\widetilde{\Lambda}^n \widetilde{\Lambda}^* \widetilde{\Lambda}^D - \widetilde{\Lambda}^D \widetilde{\Lambda}^n \widetilde{\Lambda}^* \right) \widetilde{\Lambda}^k \geq \widetilde{0}$$

Therefore $\widetilde{\Lambda}_2^n \widetilde{\Lambda}_3^* \cong \widetilde{0}$ (by remark 3.14)

Since $\tilde{\lambda} \not\cong \tilde{0}$ and $\widetilde{\Lambda}_2 \cong \tilde{0}$, we have $\widetilde{\Lambda} \tilde{v}_{f_{G(e)}} \cong \tilde{\lambda} \tilde{v}_{f_{G(e)}}$ and $\widetilde{\Lambda}^* \tilde{v}_{f_{G(e)}} \cong \overline{\tilde{\lambda}} \tilde{v}_{f_{G(e)}}$

Hence $(\widetilde{\Lambda} - \tilde{\lambda}) \tilde{v}_{f_{G(e)}} \cong \widetilde{0}$ and $\left(\widetilde{\Lambda}^* - \overline{\tilde{\lambda}} \right) \tilde{v}_{f_{G(e)}} \cong \widetilde{0}$.

Lemma 3.16. Assume $\widetilde{\Lambda}$ be FSn-QDI-hyponormal operator on fuzzy soft Hilbert space $\widetilde{H}$ if $\tilde{\sigma}_{jp}\{\widetilde{0}\} \cong \tilde{\sigma}_p\{\widetilde{0}\}$ and $(\widetilde{\Lambda} - \tilde{\lambda}) \tilde{v}_{f_{G(e)}} \cong \widetilde{0}, (\widetilde{\Lambda} - \tilde{\mu}) \tilde{u}_{f_{G(e)}} \cong \widetilde{0}, \tilde{\lambda} \not\cong \tilde{\mu}$ then $\left\langle \tilde{v}_{f_{G(e)}}, \tilde{u}_{f_{G(e)}} \right\rangle \cong \widetilde{0}$.

Proof: assume $\widetilde{\Lambda}$ be FSn-QDI-hyponormal operator, then

$$\widetilde{\Lambda}^{*k} \left(\widetilde{\Lambda}^n \widetilde{\Lambda}^* \widetilde{\Lambda}^D - \widetilde{\Lambda}^D \widetilde{\Lambda}^n \widetilde{\Lambda}^* \right) \widetilde{\Lambda}^k \geq \widetilde{0}$$

Since $(\widetilde{\Lambda} - \tilde{\lambda}) \tilde{v}_{f_{G(e)}} \cong \widetilde{0}$ and $\left(\widetilde{\Lambda}^* - \overline{\tilde{\lambda}} \right) \tilde{v}_{f_{G(e)}} \cong \widetilde{0}$ for $\tilde{v}_{f_{G(e)}} \not\cong \widetilde{0} \in \widetilde{H}$ (by lemma 3.14), by definition (2.15), (2.16) and by the equation $\tilde{\sigma}_{jp}\{\widetilde{0}\} \cong \tilde{\sigma}_p\{\widetilde{0}\}$ and assume that $\tilde{\mu} \not\cong \tilde{0}$, we have $(\widetilde{\Lambda} - \tilde{\mu}) \tilde{u}_{f_{G(e)}} \cong \tilde{0}$ (by lemma 3.15), we have,

$$\tilde{\mu} \left\langle \tilde{v}_{f_{G(e)}}, \tilde{y} \right\rangle \cong \left\langle \tilde{v}_{f_{G(e)}}, \tilde{\Lambda}^* \tilde{u}_{f_{G(e)}} \right\rangle \cong \left\langle \tilde{\Lambda} \tilde{v}_{f_{G(e)}}, \tilde{u}_{f_{G(e)}} \right\rangle \cong \left\langle \tilde{v}_{f_{G(e)}}, \tilde{u}_{f_{G(e)}} \right\rangle$$

Since $\tilde{\lambda} \not\cong \tilde{\mu}$, Hence $\left\langle \tilde{v}_{f_{G(e)}}, \tilde{u}_{f_{G(e)}} \right\rangle \cong \widetilde{0}$.

4. Conclusions

In this paper, a modest attempt is made to present and study a certain class of fuzzy soft operators defined on FS-quasi-hyponrmal operator. A novel class of fuzzy soft Drazin inverse named fuzzy soft n-quasi Drazin inverse hyponormal operator is identified and presented. They proved the identical condition for this considered fuzzy operator. Moreover, the analytic and algebraic merits of this new operator are examined. The direct sum and tensor product and the Fuzzy Soft spectrum and Fuzzy Soft approximate point spectrum for this new fuzzy operator are also discussed.

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