Fixed Point Theorems On 4-Dimensional Ball Metric Spaces And Their Applications

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We introduce the concept of B_4-metric space, also known as 4-dimensional ball metric space. Which is natural extension of metric spaces, b-metric spaces and S-metric spaces. We establish unique Fixed point theorems for a self-map on a complete 4-dimensional ball metric space with suitable contractive conditions. We also illustrate their applications. Suitable examples are provided as and when necessary.

Keywords: B_4-metric space, 4-dimensional ball metric space and Fixed point theorems.

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1. Introduction

The B_4-metric space, introduced in reference [1], emerged as a natural extension of the S-metric space initially proposed by Sedghi, Shobe, and Aliouche [2]. Within the mathematical community, these spaces have spurred significant interest and exploration, resulting in the establishment of fixed point results in references [3] and [4].

Metric spaces are generalized to three variables and are termed as S-metric spaces, which in turn are extended to four variables and are termed as B_4-metric spaces. In this paper a B_4-metric space is renamed as 4-dimensional ball metric space.

In this paper, our primary objective is to contribute to the understanding and utilisation of these metric spaces. To facilitate a clearer and more intuitive interpretation, we have chosen to rename the B_4-metric space as the 4-dimensional ball metric space. This nomenclature is in conjunction with rectangular S-metric spaces.

The concept of rectangular S-metric spaces, extending the previously established rectangular metric spaces by Branciari. It proceeds to demonstrate analogues of several well-known fixed point theorems within this unique space, thereby expanding and generalizing numerous established results in fixed point theory. This nomenclature seamlessly aligns with the conceptual framework outlined in reference [3] regarding rectangular S-metric spaces, illustrating the smooth integration of these innovative ideas into the existing body of knowledge.

While references to examples of these spaces can be found in [1], we contribute additional illustrative examples, aiming to enhance the conceptual clarity and inspire further exploration and applications of the 4-dimensional ball metric space. However, we provide some more examples in this article.

Extensions of the contraction principles can also be found in ([5–19], [20–22], [23], [24–31]) to cite a few.

2. Preliminaries

2.1. Definition

[1] Let $\Omega \neq \emptyset$ and $B_4 : \Omega^4 \to \mathbb{R}$ satisfy the following axioms: for all $t_1, t_2, t_3, t_4, \alpha \in \Omega$.

1.

$$B_4 (t_1, t_2, t_3, t_4) = 0 \text{ if and only if } t_1 = t_2 = t_3 = t_4.$$

2.

$$B_4 (t_1, t_2, t_3, t_4) \leq B_4 (t_1, t_1, t_1, \alpha) + B_4 (t_2, t_2, t_2, \alpha) +$$

$$B_4\left(t_3,t_3,t_3,\alpha\right)+B_4\left(t_4,t_4,t_4,\alpha\right).$$

Then, we say that B_4 is a 4-dimensional ball metric on Ω and the pair (Ω, B_4) is a 4-dimensional ball metric space. Various examples of 4-dimensional ball metric spaces can be found in [1]. The following are some more examples.

2.2. Example

Suppose, $\Omega = \mathbb{N} \cup \{0\}$ and define $B_4 : \Omega^4 \to \mathbb{R}^+ \cup \{0\}$ by

$$B_4\left(t_1,t_2,t_3,t_4\right)=\begin{cases}0, & \text{if } t_1=t_2=t_3=t_4: \\ t_1^2+t_2^2+t_3^2+t_4^2, & \text{otherwise.}\end{cases}$$

Then, (Ω, B_4) is a 4-dimensional ball metric space.

2.3. Example

Let $\Omega = \mathbb{N} \cup \{0\}$ and define $B_4 : \Omega^4 \to \mathbb{R}^+ \cup \{0\}$ by

$$B_4\left(t_1,t_2,t_3,t_4\right)=\begin{cases}0, & \text{if } t_1=t_2=t_3=t_4: \\ t_1+t_2+t_3+t_4, & \text{otherwise.}\end{cases}$$

Then, (Ω, B_4) is a 4-dimensional ball metric space.

2.4. Example

Let $\Omega = \mathbb{N} \cup \{0\}, \mu > 0$. Define $B_4 : \Omega^4 \to \mathbb{R}^+ \cup \{0\}$

$$\text{by } B_4\left(t_1,t_2,t_3,t_4\right)=\begin{cases}0, & \text{if } t_1=t_2=t_3=t_4: \\ \mu, & \text{otherwise,}\end{cases}$$

where $t_1, t_2, t_3, t_4 \in \Omega$. Then, (Ω, B_4) is a 4 -dimensional ball metric space.

The notions of limit, convergence, Cauchy sequence and completeness in a 4-dimensional ball metric space are given in [1] as follows:

Definition 04: Let (Ω, B_4) be a 4-dimensional ball metric space.

1. A sequence $\{t_n\}$ in Ω converges to t if $B_4\left(t_n,t_n,t_n,t\right) \to 0$, as $n \to \infty$.

   That is, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0, B_4\left(t_n,t_n,t_n,t\right) < \varepsilon$.

   We denote this by $\lim_{n\to\infty} t_n = t$ or $\lim_{n\to\infty} B_4\left(t_n, t_n,t_n,t\right)$=0.

2. A sequence $\{t_n\}$ in Ω is called a Cauchy sequence if $B_4\left(t_n,t_n,t_n,t_m\right) \to 0$, as $n, m \to \infty$.

   That is, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $n, m \geq n_0, B_4\left(t_n,t_n,t_n,t_m\right) < \varepsilon$.

3. A 4-dimensional ball metric space (Ω, B_4) is called complete if every Cauchy sequence in Ω is convergent.

We now state a few lemmas, which we use in our further development.

2.5. Lemma

[1] Let (Ω, B_4) be a 4-dimensional ball metric space. Then, $B_4\left(t_1,t_2,t_2,t_2\right) = B_4\left(t_2,t_2,t_2,t_1\right)$, for all $t_1, t_2 \in \Omega$.

2.6. Lemma

[1] $t_m \to t$ if and only if $B_4\left(t,t,t,t_m\right) \to 0$, as $m \to \infty$.

2.7. Lemma

[1] If $t_m \to \eta_1$ and $t_m \to \eta_2 \Rightarrow \eta_1 = \eta_2$.

2.8. Lemma

[1] $t_m \to t \Rightarrow \{t_m\}$ is a Cauchy Sequence.

3. Main results for 4-dimensional ball metric spaces

First, we prove the following lemma and use it in our main result.

3.1. Lemma

Let $\Omega \neq \emptyset$ and $B_4 : \Omega^4 \to \mathbb{R}^+ \cup \{0\}$ be a 4-dimensional ball metric space on Ω. Then, $B_4(\alpha,\beta,\beta,\beta) \leq B_4(\alpha,\gamma,\gamma,\gamma) + 3B_4(\gamma,\beta,\beta,\beta)$, for all $\alpha, \beta, \gamma \in \Omega$.

Proof: Suppose, (Ω, B_4) is a 4-dimensional ball metric space.

Replacing t_1 by α, t_2, t_3, t_4 by β and α by γ in definition 2.1 (ii), we get, $B_4(\alpha,\beta,\beta,\beta) \leq B_4(\alpha,\alpha,\alpha,\gamma) + B_4(\beta,\beta,\beta,\gamma) + B_4(\beta,\beta,\beta,\gamma) + B_4(\beta,\beta,\beta,\gamma)$.

Therefore, $B_4(\alpha,\beta,\beta,\beta) \leq B_4(\alpha,\alpha,\alpha,\gamma) + 3B_4(\beta,\beta, \beta,\gamma)$.

Therefore, $B_4(\alpha,\beta,\beta,\beta) \leq B_4(\alpha,\gamma,\gamma,\gamma) + 3B_4(\gamma,\beta, \beta,\beta)$. (by Lemma 2.5)

Now we state and prove our main theorem on 4-dimensional ball metric spaces.

3.2. Theorem

Suppose, (Ω, B_4) is a complete 4-dimensional ball metric space and $\mathrm{T} : \Omega \to \Omega$ is a map. Suppose, $0 \leq k < \frac{1}{3}$ is such that for all $t_1, t_2, t_3, t_4 \in \Omega$,

$$B_4\left(\mathrm{T}t_1,\mathrm{T}t_2,\mathrm{T}t_3,\mathrm{T}t_4\right) \leq kB_4\left(t_1,t_2,t_3,t_4\right). \quad (3.2.1)$$

Then, T has a unique fixed point.

Proof: For $t_1, t_2 \in \Omega$, from (3.2.1), taking $t_2 = t_3 = t_4$, we have $B_4\left(\mathrm{T}t_1,\mathrm{T}t_2,\mathrm{T}t_2,\mathrm{T}t_2\right) \leq kB_4\left(t_1,t_2,t_2,t_2\right)$.

Let $t_0 \in \Omega$. Define the sequence $\{t_n\}$ by $t_n = \mathrm{T}^n t_0$, for $n = 1, 2, 3, \ldots$.

Then, clearly $t_{n+1} = \mathrm{T}^n t_n$.

$$B_4\left(t_n,t_n,t_n,t_{n+1}\right) = B_4\left(\mathrm{T}t_{n-1},Tt_{n-1},\mathrm{T}t_{n-1},\mathrm{T}t_n\right) \leq$$

$$kB_4\left(t_{n-1},t_{n-1},t_{n-1},t_n\right)$$

Write $s_n = B_4\left(t_n,t_n,t_n,t_{n+1}\right)$. (3.2.2)

We have $s_n \leq ks_{n-1} \leq k^2 s_{n-2}$.
Therefore, $s_n \leq k^n s_0, \forall n \in N$. (3.2.3)
This shows that $s_n \to 0$, as $n \to \infty$. Suppose, $m > n$.
By using Definition 2.1, for $t_{n+1}, t_{n+2}, \ldots, t_{m-1}$, we have

$$B_4(t_n, t_m, t_m, t_m) \leq \left\{ \begin{array}{c} B_4(t_n, t_n, t_n, t_{n+1}) + B_4(t_m, t_m, t_m, t_{n+1}) \\ +B_4(t_m, t_m, t_m, t_{n+1}) + B_4(t_m, t_m, t_m, t_{n+1}) \end{array} \right\}$$
$$= B_4(t_n, t_n, t_n, t_{n+1}) + 3B_4(t_m, t_m, t_m, t_{n+1})$$
$$= s_n + 3B_4(t_m, t_m, t_m, t_{n+1})$$
$$\leq s_n + 3s_{n+1} + 3^2 B_4(t_m, t_m, t_m, t_{n+2})$$
$$\leq s_n + 3s_{n+1} + 3^2 s_{n+2} + 3^3 B_4(t_m, t_m, t_m, t_{n+3})$$
$$\leq s_n + 3s_{n+1} + 3^2 s_{n+2} + 3^3 s_{n+3} + \cdots + 3^{m-n-1} s_{m-1}$$

(by Lemma 3.1). (3.2.4)
From (3.2.3) and (3.2.4), we have

$$B_4(t_n, t_m, t_m, t_m) \leq s_n + 3ks_{n+1} + 3^2k^2 s_{n+2} + 3^3k^3 s_{n+3} + \cdots + 3^{m-n-1}k^{m-n-1} s_{m-1}$$
$$\leq s_n \left(1 + 3k + (3k)^2 + \cdots + (3k)^{m-n-1}\right)$$
$$= s_n \left(1 + \mu + \mu^2 + \cdots + \mu^{m-n-1}\right), \text{ where } \mu = 3k$$
$$\leq s_n \left(\frac{1}{1-\mu}\right) (\text{ since } \mu < 1, \text{ by hypothesis })$$
$$\leq k^n s_0 \left(\frac{1}{1-\mu}\right) \to 0, \text{ as } n \to \infty, \text{ since } k < \frac{1}{3}.$$

Hence, t_n is a Cauchy sequence.
Since, Ω is complete, there exists $t^* \in \Omega$ such that $t_n \to t^*$. (3.2.5)
Now

$$B_4(t_{n+1}, Tt^*, Tt^*, Tt^* = B_4(Tt_n, Tt^*, Tt^*, Tt^*)$$
$$\leq kB_4(t_n, t^*, t^*, t^*)$$
$$\to 0, \text{ as } n \to \infty \ (by\ 3.2.5)$$

Therefore, $t_{n+1} \to Tt^*$.
Therefore, $Tt^* = t^*$. (by Lemma 2.7)
Therefore, t^* is a fixed point of T.
Suppose, t^{**} is a fixed point of T.
Then, $B_4(Tt^*, Tt^{**}, Tt^{**}, Tt^{**}) \leq kB_4(t^*, t^{**}, t^{**}, t^{**})$.
Therefore, $B_4(t^*, t^{**}, t^{**}, t^{**}) \leq kB_4(t^*, t^{**}, t^{**}, t^{**})$.
Therefore, $B_4(t^*, t^{**}, t^{**}, t^{**}) = 0$.
Therefore, $t^* = t^{**}$. (by Definition 2.1)
Thus, T has a unique fixed point, namely t^*.

4. Applications

In this section we obtain applications of the Theorem 3.2.

4.1. Theorem

Suppose, (Ω, B_4) is a complete 4-dimensional ball metric space and $T : \Omega \to \Omega$ is a map. Suppose, $0 \leq k < 0.1$ is such that for all $t_1, t_2, t_3, t_4 \in \Omega$.

$$B_4(Tt_1, Tt_2, Tt_3, Tt_4) \leq k \left\{ \begin{array}{c} B_4(t_1, Tt_1, Tt_1, Tt_1) + B_4(t_2, Tt_2, Tt_2, Tt_2) \\ +B_4(t_3, Tt_3, Tt_3, Tt_3) + B_4(t_4, Tt_4, Tt_4, Tt_4) \end{array} \right\}. \quad (4.1.1)$$

Then, T has a unique fixed point.

Proof: From (4.1.1), taking $t_2 = t_3 = t_4$, we have $B_4(Tt_1, Tt_2, Tt_2, Tt_2) \leq k\{B_4(t_1, Tt_1, Tt_1, Tt_1) + 3B_4(t_2, Tt_2, Tt_2, Tt_2)\}$.

Let $t_0 \in \Omega$ and define the sequence $\{t_n\}$ by $t_{n+1} = T^n t_0$ for $n = 0, 1, 2, \ldots$.

We have
$$B_4(t_n, t_n, t_n, t_{n+1}) \leq k \left(\begin{array}{c} B_4(t_{n-1}, t_{n-1}, t_{n-1}, t_n) + B_4(t_{n-1}, t_{n-1}, t_{n-1}, t_n) \\ +B_4(t_{n-1}, t_{n-1}, t_{n-1}, t_n) + B_4(t_n, t_n, t_n, t_{n+1}) \end{array} \right)$$
so that, $B_4(t_n, t_n, t_n, t_{n+1}) \leq \frac{3k}{1-k} B_4(t_{n-1}, t_{n-1}, t_{n-1}, t_n)$.

Since, $\frac{3k}{1-k} < \frac{1}{3}$, by Theorem 3.2, the result follows.
Write $s_n = B_4(t_n, t_n, t_n, t_{n+1})$.
we have $s_n \leq ks_{n-1} \leq k^2 s_{n-2}$.
Therefore, $s_n \leq k^n s_0, \forall n \in \mathbb{N}$, we have for all $n, m \in \mathbb{N}$ with $n \neq m, t_n \neq t_m$.
By repeated use of (ii) in Definition 2.1,

$$B_4(t_n, t_m, t_m, t_m) \leq \left\{ \begin{array}{c} B_4(t_n, t_n, t_n, t_{n+1}) + B_4(t_m, t_m, t_m, t_{n+1}) \\ B_4(t_m, t_m, t_m, t_{n+1}) + B_4(t_m, t_m, t_m, t_{n+1}) \end{array} \right\}$$
$$= B_4(t_n, t_n, t_n, t_{n+1}) + 3B_4(t_m, t_m, t_m, t_{n+1})$$
$$\leq s_n + 3B_4(t_m, t_m, t_m, t_{n+2})$$
$$\leq s_n + 3s_{n+1} + 3^2 B_4(t_m, t_m, t_m, t_{n+2})$$
$$\leq s_n + 3s_{n+1} + 3^2 s_{n+2} + 3^3 s_{n+3} + \cdots + 3^{m-n-1} s_{m-1}$$

We have,

$$B_4(t_n, t_m, t_m, t_m) \leq k^n s_0 + 3k^{n+1} s_0 + 3^2 k^{n+2} s_0 + \cdots + 3^{m-1} k^{m-1} s_0$$
$$\leq \left(k^n + 3k^{n+1} + 3^2 k^{n+2} + \cdots + 3^{m-1} k^{m-1}\right) s_0$$
$$\leq k^n \left(1 + 3k + (3k)^2 + (3k)^3 + \cdots + (3k)^{m-n-1}\right) s_0$$
$$\leq k^n \left(1 + \mu + (\mu)^2 + (\mu)^3 + \cdots + (\mu)^{m-n-1}\right) s_0$$
$$\leq \frac{k^n}{(1-\mu)} s_0 \to 0, \text{ as } n \to \infty.$$

Therefore, $\{t_n\}$ is Cauchy sequence.
There exists, $t^* \in \Omega$ such that $t_n \to t^*$, since Ω is complete.
Now $B_4(t_{n+1}, Tt^*, Tt^*, Tt^*) = B_4(Tt_n, Tt^*, Tt^*, Tt^*)$

$$\leq B_4(t_n, t^*, t^*, t^*) \to 0, \text{ as } n \to \infty.$$

Therefore, $t_{n+1} \to \mathrm{T}t^*$.

Therefore, $\mathrm{T}t^* = t^*$.

Therefore, t^* is a fixed point of T.

$$B_4\left(\mathrm{T}t^*, \mathrm{T}t^{**}, \mathrm{T}t^{**}, \mathrm{T}t^{**}\right)$$
$$\leq k \left\{ \begin{array}{c} B_4\left(t^*, \mathrm{T}t^*, \mathrm{T}t^*, \mathrm{T}t^*\right) + B_4\left(t^{**}, \mathrm{T}t^{**}, \mathrm{T}t^{**}, \mathrm{T}t^{**}\right) \\ +B_4\left(t^{**}, \mathrm{T}t^{**}, \mathrm{T}t^{**}, \mathrm{T}t^{**}\right) + B_4\left(t^{**}, \mathrm{T}t^{**}, \mathrm{T}t^{**}, \mathrm{T}t^{**}\right) \end{array} \right\}$$
$$\leq k \left\{ \begin{array}{c} B_4\left(t^*, t^*, t^*, t^*\right) + B_4\left(t^{**}, t^{**}, t^{**}, t^{**}\right) \\ +B_4\left(t^{**}, t^{**}, t^{**}, t^{**}\right) + B_4\left(t^{**}, t^{**}, t^{**}, t^{**}\right) \end{array} \right\}.$$

Therefore, $B_4\left(t^*, t^{**}, t^{**}, t^{**}\right) \leq kB_4\left(t^*, t^{**}, t^{**}, t^{**}\right)$.

Therefore, $B_4\left(t^*, t^{**}, t^{**}, t^{**}\right) \leq 0$.

Therefore, $t^* = t^{**}$.

Thus, T has a unique fixed point, namely t^*.

4.2. Theorem

Suppose, (Ω, B_4) is a complete 4-dimensional ball metric space and $\mathrm{T} : \Omega \to \Omega$ is a map.

Suppose, the real numbers g_1, g_2, g_3, g_4 are such that $0 \leq g_1 < \frac{1}{3}, 0 \leq g_2 < \frac{1}{4}, 0 \leq g_3 < \frac{1}{4}, 0 \leq g_4 < \frac{1}{4}$.

Write $\delta = \max\left\{g_1, \frac{g_2}{1-g_2}, \frac{g_3}{1-g_3}, \frac{g_4}{1-g_4}\right\}$.

Assume that for all $t_1, t_2, t_3, t_4 \in \Omega$, $B_4(\mathrm{T}t_1, \mathrm{T}t_2, \mathrm{T}t_3, \mathrm{T}t_4) \leq \frac{\delta}{4} B_4\left(t_1, t_2, t_3, t_4\right) + 3\frac{\delta}{4} B_4\left(t_1, t_1, t_1, \mathrm{T}t_1\right)$. (4.2.1)

Then, T has a unique fixed point.

Proof: From (4.2.1), taking $t_2 = t_3 = t_4$, we have

$$B_4\left(\mathrm{T}t_1, \mathrm{T}t_2, \mathrm{T}t_2, \mathrm{T}t_2\right) \leq \frac{\delta}{4} B_4\left(t_1, t_2, t_2, t_2\right) + \frac{3\delta}{4} B_4\left(t_1, t_1, t_1, \mathrm{T}t_1\right).$$

Let $t_0 \in \Omega$ and define the sequence $\{t_n\}$ by $t_{n+1} = T^n t_0$.

Then,

$$B_4\left(t_{n+1}, t_n, t_n, t_n\right) = B_4\left(\mathrm{T}t_n, \mathrm{T}t_{n-1}, \mathrm{T}t_{n-1}, \mathrm{T}t_{n-1}\right)$$
$$\leq \frac{\delta}{4} B_4\left(t_n, t_{n-1}, t_{n-1}, t_{n-1}\right) + \frac{3\delta}{4} B_4\left(t_n, t_{n-1}, t_{n-1}, t_{n-1}\right)$$
$$= \delta B_4\left(t_n, t_{n-1}, t_{n-1}, t_{n-1}\right).$$

Setting $s_n = B_4\left(t_n, t_n, t_n, t_{n+1}\right)$, then

$$s_n \leq ks_{n-1}, s_{n-1} \leq ks_{n-2} \Rightarrow s_n \leq k^n s_0, \forall n \in \mathbb{N}.$$

By repeated use of (ii) in Definition 2.1,

$$\begin{aligned} B_4\left(t_n, t_m, t_m, t_m\right) \leq & B_4\left(t_n, t_n, t_n, t_{n+1}\right) + B_4\left(t_m, t_m, t_m, t_{n+1}\right) \\ & + B_4\left(t_m, t_m, t_m, t_{n+1}\right) + B_4\left(t_m, t_m, t_m, t_{n+1}\right) \\ = & B_4\left(t_n, t_n, t_n, t_{n+1}\right) + 3B_4\left(t_m, t_m, t_m, t_{n+1}\right) \\ = & s_n + 3B_4\left(t_m, t_m, t_m, t_{n+1}\right) \\ \leq & s_n + 3s_{n+1} + 3^2 B_4\left(t_m, t_m, t_m, t_{n+2}\right) \\ \leq & s_n + 3s_{n+1} + 3^2 s_{n+2} + 3^3 B_4\left(t_m, t_m, t_m, t_{n+3}\right) \\ \leq & s_n + 3s_{n+1} + 3^2 s_{n+2} + 3^3 s_{n+3} + \cdots + 3^{m-1} s_m \\ \leq & s_n + 3s_{n+1} + 3^2 s_{n+2} + 3^3 s_{n+3} + \cdots \end{aligned}$$

We have $B_4\left(t_n, t_m, t_m, t_m\right)$

$$\leq k^n s_0 + 3k^{n+1} s_0 + 3^2 k^{n+2} s_0 + \cdots + 3^{m-1} k^{m-1} s_0$$
$$\leq \left(k^n + 3k^{n+1} + 3^2 k^{n+2} + \cdots + 3^{m-1} k^{m-1}\right) s_0$$
$$\leq k^n \left(1 + 3k + (3k)^2 + (3k)^3 + \cdots + (3k)^{m-n-1}\right) s_0$$
$$\leq k^n \left(1 + 3k + (3k)^2 + (3k)^3 + (3k)^4 + \cdots\right) s_0$$
$$\leq \frac{k^n}{(1-3k)} s_0 \to 0, \text{ as } n \to \infty.$$

Therefore, $\{t_n\}$ is Cauchy sequence.

There exists, $t^* \in \Omega$ such that $t_n \to t^*$, since Ω is complete.

Now $B_4\left(t_{n+1}, Tt^*, Tt^*, Tt^*\right) = B_4\left(Tt_n, Tt^*, Tt^*, Tt^*\right)$

$$\leq B_4\left(t_n, t^*, t^*, t^*\right) \to 0, \text{as } n \to \infty.$$

Therefore, $t_{n+1} \to \mathrm{T}t^*$.

Therefore, $\mathrm{T}t^* = t^*$.

Therefore, t^* is a fixed point of T.

Suppose, t^{**} is a fixed point of T. Then,

$$B_4\left(\mathrm{T}t^*, \mathrm{T}t^{**}, \mathrm{T}t^{**}, \mathrm{T}t^{**}\right) \leq \frac{\delta}{4} B_4\left(t^*, t^{**}, t^{**}, t^{**}\right) + \frac{3\delta}{4} B_4\left(t^*, t^{**}, t^{**}, t^{**}\right).$$

Therefore, $B_4\left(t^*, t^{**}, t^{**}, t^{**}\right) \leq \delta B_4\left(t^*, t^{**}, t^{**}, t^{**}\right)$

Therefore, $B_4\left(t^*, t^{**}, t^{**}, t^{**}\right) = 0$. (since $\delta > 1$)

Therefore, $t^* = t^{**}$.

Thus, T has a unique fixed point, namely t^*.

4.3. Theorem

Suppose, (Ω, B_4) is a complete 4-dimensional ball metric space and $\mathrm{T} : \Omega \to \Omega$ is a map. Suppose, g_1, g_2, g_3, g_4 are such that $0 \leq g_1 < \frac{1}{4}, 0 \leq g_2 < \frac{1}{5}, 0 \leq g_3 < \frac{1}{5}, 0 \leq g_4 < \frac{1}{5}$. Write $\delta = \max\left\{g_1, \frac{g_2}{1-g_2}, \frac{g_3}{1-g_3}, \frac{g_4}{1-g_4}\right\}$. Suppose, for all $t_1, t_2, t_3, t_4 \in \Omega$, $B_4\left(\mathrm{T}t_1, \mathrm{T}t_2, \mathrm{T}t_3, \mathrm{T}t_4\right) \leq \frac{\delta}{4} B_4\left(t_1, t_2, t_3, t_4\right) + \frac{3\delta}{3} B_4\left(t_1, t_1, t_1, \mathrm{T}t_1\right.$

Then, T has a unique fixed point.

Proof: From (4.3.1), taking $t_2 = t_3 = t_4$, we have

$$B_4\left(\mathrm{T}t_1, \mathrm{T}t_2, \mathrm{T}t_2, \mathrm{T}t_2\right) \leq \frac{\delta}{4} B_4\left(t_1, t_2, t_2, t_2\right) + \frac{3\delta}{3} B_4\left(t_1, t_1, t_1, \mathrm{T}t_1\right)$$

Let $t_0 \in \Omega$ and define the sequence $\{t_n\}$ by $t_{n+1} = \mathrm{T}t_n$.

Then,

$$B_4\left(t_{n+1}, t_n, t_n, t_n\right) = B_4\left(\mathrm{T}t_n, \mathrm{T}t_{n-1}, \mathrm{T}t_{n-1}, \mathrm{T}t_{n-1}\right)$$
$$\leq \frac{\delta}{4} B_4\left(t_n, t_{n-1}, t_{n-1}, t_{n-1}\right) + \frac{3\delta}{3} B_4\left(t_n, t_{n-1}, t_{n-1}, t_{n-1}\right)$$
$$= \frac{5}{4} \delta B_4\left(t_n, t_{n-1}, t_{n-1}, t_{n-1}\right).$$

Since, $\frac{5}{4}\delta < \frac{1}{3}$, by Theorem 3.2 it follows that, $\{t_n\}$ is Cauchy sequence and Ω is complete, there exists $t^* \in \Omega$ such that $t_n \to t^*$

Therefore, T has a unique fixed point, namely t^*.

Conclusion

In this paper, we **introduce** the notation of B_4**-metric space**, or the 4-dimensional ball metric space. We **establish unique Fixed point theorems** for a self-map on a complete 4-dimensional ball metric space and **illustrate** their **applications with supporting examples**. This work **contributes** to the **understanding** and application of 4-dimensional ball metric space **in mathematical analysis** and its allied areas.

Further scope

The possibility of extending the results of this paper to n-dimensional ball metric spaces are under active investigation.

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