The New Extended Direct Algebraic Method For Modified KdV-Zakharov-Kuznetsov

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Received: June 22, 2023; Accepted: Nov. 12, 2023

Finding traveling wave (TW) solutions for nonlinear equations has always been one of the most important concerns of researchers in various mathematics, physics, and engineering fields. In this paper, we employ a new extended direct algebraic (NEDA) technique to study the modified KdV-Zakharov-Kuznetsov (mKdV-ZK) equation. In the framework of this technique, various forms of analysis solutions for the equation are obtained, which have many applications in the field of electric and magnetic fields. The correctness of all the solutions introduced in this paper has been checked after their direct replacement in the equation. Moreover, numerical simulations corresponding to some of these analytical solutions are included in the paper.

Keywords: Travelling wave solution, The modified KdV-Zakharov-Kuznetsov equation; New extended direct algebraic method; Analytical solutions; Numerical simulations

http://dx.doi.org/10.6180/jase.202410_27(10).0013

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1. Introduction

In the past decades, solving nonlinear models has gained a lot of attention among scholars to model and describe applied problems in science and technology such as plasma physics [1], electrical transmission lines [2], ion-acoustic waves in plasma [3], financial mathematics [4] and many others applications [5–8]. This high importance for this group of equations has led to the presentation of many techniques for solving such problems in the recent literature. Some of these methods are the improved generalized Riccati equation mapping technique [9], the new extended direct algebraic technique [10], the new Sardar sub-ODE technique [11], the modified Kudryashov technique [12], the generalized Kudryashov technique [13], the improved Bernoulli sub-equation function technique [14], the generalized exponential rational function technique [15], and the extended ϕ^6-model expansion technique [16], auxiliary equation technique [17], the sine-Gordon expansion technique [18], the Riccati-Bernoulli Sub-ODE technique [19], the modified Khater technique [20], the improved F-expansion technique [21], the generalized $\exp(-\varphi(\xi))$-expansion (GEE) technique [22], the Exp-function technique [23], and so on [24–46].

In this work, we consider the mKdV-ZK equation as [47–53]:

$$\chi_t + 2\chi^2\chi_x + \chi_{xxx} + \chi_{xyy} + \chi_{xzz} = 0 \tag{1}$$

Where $\chi = \chi(x, y, z, t)$ and χ is the amplitude of the relevant wave mode. This equation is one the most widely used equation in plasma, which is used to describe the evolution of ion-acoustic perturbations in magnetized plasma with two negative ion components of different temperatures [47, 48]. Due to its many applications, various techniques and approaches have been used in solving this equation. For example, the authors of [49] have utilized the Lie group analysis to study the similarity reduction and exact solutions of the mKdV-ZK equation. The work of [50] consists of applying two integration techniques, including the first integral technique and the functional variable technique, in solving model (1). Khan and Akbar [51] have used the modified simple equation (MSE) technique to discover new solutions to Eq. (1) and proposed several solitary wave solutions for the model. The variable-separated ODE method is the integration method used in [52] to solve the conformable fractional version of the model. Taking and enhanced (G'/G)-expansion technique, the authors of [53] have derived abundant TW solutions for the model (1). In the work of [54], the ansatz equation has been extended in the modified, extended direct algebraic technique to find optical wave solutions of the three-dimensional form of the model.

In this article, we try to provide an application of the NEDA technique in solving model (1). To this end, the presentation framework for the article is outlined as follows: Then, mathematical analysis with the NEDA method for the main model in Sec. 2. In Sec. 4, we give some figures that show the TW solutions of Eq. (1). The last section of the article in Sec. 3 is devoted to conclusions remarks.

2. Mathematical analysis for the main model

In this section, we will apply the NEDA technique to construct many new and more general TW solutions for the mKdV-ZK equation [55].

We use the wave transformation as:

$$\chi(x,y,z,t)=\wp(o), \quad o=\kappa_1 x+\kappa_2 y+\kappa_3 z-\omega t \tag{2}$$

To reduce Eq. (1) to the nonlinear ODE below:

$$\omega\wp'+\delta\kappa_1\wp^2\wp'+\left(\kappa_1^3+\kappa_1\kappa_2^2+\kappa_1\kappa_3^2\right)\wp'''=0 \tag{3}$$

where $\kappa_1,\kappa_2,\kappa_3$ and ω are arbitrary constants. Balancing the nonlinear term $\wp^3$ with the highest order derivative term $\wp'''$ in Eq. (3), we have:

$$\wp(o)=b_0+b_1G(o) \tag{4}$$

where b_0,b_1 are constant coefficients to be determined later. Substituting Eq. (4) along with

$$G'(o)=\mathrm{Ln}(\Theta)\left(a+bG(o)+cG^2(o)\right), \quad \Theta\neq 0,1 \tag{5}$$

Where a,b and c are constants, into Eq. (3), collecting the coefficients of $G^l(o)$ and equating them to zero. Coefficients of $G(o)$ as follows:

$$\begin{aligned}
G^0 :&ab_1\ln(\Theta)\left(\omega+\delta\kappa_1b_0^2+\left(b^2+2ca\right)\left(\kappa_1^3+\kappa_1\kappa_2^2+\kappa_1\kappa_3^2\right)\ln(\Theta)\right)\\
&=0\\
G^1 :&b_1\ln(\Theta)\left(\omega b+\delta b\kappa_1b_0^2+2\delta\kappa_1b_0b_1a+6abc\left(\kappa_1^3+\kappa_1\kappa_2^2+\kappa_1\kappa_3^2\right)\right.\\
&\left.\ln^2(\Theta)+b\left(b^2+2ca\right)\left(\kappa_1^3+\kappa_1\kappa_2^2+\kappa_1\kappa_3^2\right)\ln^2(\Theta)\right)=0\\
G^2 :&b_1\ln(\Theta)(\omega c+\delta c\kappa_1b_0^2+2\delta b\kappa_1b_0b_1+\delta a\kappa_1b_1^2+\\
&6ac^2\left(\kappa_1^3+\kappa_1\kappa_2^2+\kappa_1\kappa_3^2\right)\ln^2(\Theta)\\
&+6b^2c\left(\kappa_1^3+\kappa_1\kappa_2^2+\kappa_1\kappa_3^2\right)\ln^2(\Theta)+\\
&c\left(b^2+2ca\right)\left(\kappa_1^3+\kappa_1\kappa_2^2+\kappa_1\kappa_3^2\right)\ln^2(\Theta)=0\\
G^3 :&b_1\ln(\Theta)\left(2\delta c\kappa_1b_0b_1+\delta b\kappa_1b_1^2+\right.\\
&\left.12bc^2\left(\kappa_1^3+\kappa_1\kappa_2^2+\kappa_1\kappa_3^2\right)\ln^2(\Theta)\right)=0\\
G^4 :&\ cb_1\ln(\Theta)\left(\delta\kappa_1b_1^2+6c^2\left(\kappa_1^3+\kappa_1\kappa_2^2+\kappa_1\kappa_3^2\right)\ln^2(\Theta)\right)=0
\end{aligned}$$

By solving the above algebraic equations using Maple, we have:

$$\begin{aligned}
b_0&=\mp\frac{1}{2}b\ln(\Theta)\sqrt{-\frac{6\left(\kappa_1^2+6\kappa_2^2+6\kappa_3^2\right)}{\delta}},\\
b_1&=\pm c\ln(\Theta)\sqrt{-\frac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}},\\
\omega&=\frac{(b^2-4ca)}{2}\kappa_1\ln^2(\Theta)\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right).
\end{aligned} \tag{6}$$

The solutions of Eq. (1) according to these values

When $\Upsilon=b^2-4ac<0$ and $c\neq 0$,

$$\begin{aligned}
\chi_1^{\pm}&=\pm\ln(\Theta)\sqrt{-\frac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left[b-\frac{\sqrt{-\Upsilon}}{2}\tan_\Theta\left(\frac{\sqrt{-\Upsilon}}{2}o\right)\right].\\
\chi_2^{\pm}&=\pm\ln(\Theta)\sqrt{-\frac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left[b+\frac{\sqrt{-\Upsilon}}{2}\cot_\Theta\left(\frac{\sqrt{-\Upsilon}}{2}o\right)\right]\\
\chi_3^{\pm}&=\pm\ln(\Theta)\sqrt{-\frac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\\
&\quad\left[b-\frac{\sqrt{-\Upsilon}}{2}\left(\tan_\Theta(\sqrt{-\Upsilon}o)\pm\sqrt{pq}\sec_\Theta(\sqrt{-\Upsilon}o)\right)\right]\\
\chi_4^{\pm}&=\pm\ln(\Theta)\sqrt{-\frac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\\
&\quad\left[b-\frac{\sqrt{-\Upsilon}}{2}\left(-\cot_\Theta(\sqrt{-\Upsilon}o)\pm\sqrt{pq}\csc_\Theta(\sqrt{-\Upsilon}o)\right)\right]\\
\chi_5^{\pm}&=\ln(\Theta)\sqrt{-\frac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\\
&\quad\left[b-\frac{\sqrt{-\Upsilon}}{4}\left(\tan_\Theta\left(\frac{\sqrt{-\Upsilon}}{4}o\right)-\cot_\Theta\left(\frac{\sqrt{-\Upsilon}}{4}o\right)\right)\right],
\end{aligned}$$

where

$$o=\kappa_1x+\kappa_2y+\kappa_3z-\left(\frac{1}{2}\kappa_1\Upsilon\ln^2(\Theta)\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)\right)t.$$

When $\Upsilon=b^2-4ac>0$ and $c\neq 0$,

$$\chi_6^{\pm} = \pm\ln(\Theta)\sqrt{-\tfrac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left[b+\tfrac{\sqrt{\mathrm{Y}}}{2}\tanh_\Theta\left(\tfrac{\sqrt{\mathrm{Y}}}{2}o\right)\right],$$

$$\chi_7^{\pm} = \pm\ln(\Theta)\sqrt{-\tfrac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left[b+\tfrac{\sqrt{\mathrm{Y}}}{2}\coth_\Theta\left(\tfrac{\sqrt{\mathrm{Y}}}{2}o\right)\right],$$

$$\chi_8^{\pm} = \pm\ln(\Theta)\sqrt{-\frac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left[b-\frac{\sqrt{\mathrm{Y}}}{2}\left(-\tanh_\Theta(\sqrt{\mathrm{Y}}o)\pm i\sqrt{pq}\,\mathrm{sech}_\Theta(\sqrt{\mathrm{Y}}o)\right)\right]$$

$$\chi_9^{\pm} = \pm\ln(\Theta)\sqrt{-\frac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left[\frac{1}{2}b-\frac{\sqrt{\mathrm{Y}}}{2}\left(-\coth_\Theta(\sqrt{\mathrm{Y}}o)\pm\sqrt{pq}\,\mathrm{csch}_\Theta(\sqrt{\mathrm{Y}}o)\right)\right]$$

$$\chi_{10}^{\pm} = \pm\ln(\Theta)\sqrt{-\frac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left[b+\frac{\sqrt{\mathrm{Y}}}{4}\left(\tanh_\Theta\left(\frac{\sqrt{\mathrm{Y}}}{4}o\right)+\coth_\Theta\left(\frac{\sqrt{\mathrm{Y}}}{4}o\right)\right)\right]$$

where

$$o = \kappa_1 x+\kappa_2 y+\kappa_3 z-\left(\frac{1}{2}\kappa_1\mathrm{Y}\ln^2(\Theta)\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)\right)t$$

When $ac > 0$ and $b = 0$,

$$\chi_{11}^{\pm} = \pm\ln(\Omega)\sqrt{-\tfrac{6ac\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\tan_\Theta(\sqrt{ac}o)$$

$$\chi_{12}^{\pm} = \pm\ln(\Omega)\sqrt{-\tfrac{6ac\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\cot_\Theta(\sqrt{ac}o)$$

$$\chi_{13}^{\pm} = \pm\ln(\Omega)\sqrt{-\frac{6ac\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left(\tan_\Theta(2\sqrt{ac}o)\pm\sqrt{pq}\sec_\Theta(2\sqrt{ac}o)\right)$$

$$\chi_{14}^{\pm} = \pm\ln(\Omega)\sqrt{-\frac{6ac\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left(-\cot_\Theta(2\sqrt{ac}o)\pm\sqrt{pq}\csc_\Theta(2\sqrt{ac}o)\right)$$

$$\chi_{15}^{\pm} = \pm\ln(\Omega)\sqrt{-\frac{3ac\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{2\delta}}\left(\tan_\Theta\left(\frac{\sqrt{ac}}{2}o\right)-\cot_\Theta\left(\frac{\sqrt{ac}}{2}o\right)\right)$$

where

$$o = \kappa_1 x+\kappa_2 y+\kappa_3 z+\left(2ac\kappa_1\ln^2(\Omega)\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)\right)t$$

When $ac < 0$ and $b = 0$,

$$\chi_{16}^{\pm} = \pm\ln(\Omega)\sqrt{\tfrac{6ac\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\tanh_\Theta(\sqrt{-ac}o)$$

$$\chi_{17}^{\pm} = \pm\ln(\Omega)\sqrt{\tfrac{6ac\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\coth_\Theta(\sqrt{-ac}o)$$

$$\chi_{18}^{\pm} = \pm\ln(\Omega)\sqrt{\frac{6ac\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left(-\tanh_\Theta(2\sqrt{-ac}o)\pm i\sqrt{pq}\,\mathrm{sech}_\Theta(2\sqrt{-ac}o)\right)$$

$$\chi_{19}^{\pm} = \pm\ln(\Omega)\sqrt{\frac{6ac\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left(-\coth_\Theta(2\sqrt{-ac}o)\pm\sqrt{pq}\,\mathrm{csch}_\Theta(2\sqrt{-ac}o)\right)$$

$$\chi_{20}^{\pm} = \pm\ln(\Omega)\sqrt{\frac{3ac\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{2\delta}}\left(\tanh_\Theta\left(\frac{\sqrt{-ac}}{2}o\right)+\coth_\Theta\left(\frac{\sqrt{-ac}}{2}o\right)\right)$$

where

$$o = \kappa_1 x+\kappa_2 y+\kappa_3 z+\left(2ac\kappa_1\ln^2(\Omega)\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)\right)t$$

When $b = 0$ and $c = a$,

$$\chi_{21}^{\pm} = \pm a\ln(\Omega)\sqrt{-\tfrac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\tan_\Theta(ao)$$

$$\chi_{22}^{\pm} = \pm a\ln(\Omega)\sqrt{-\tfrac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\cot_\Theta(ao)$$

$$\chi_{23}^{\pm} = \pm a\ln(\Omega)\sqrt{-\frac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left(\tan_\Theta(2ao)\pm\sqrt{pq}\sec_\Theta(2ao)\right)$$

$$\chi_{24}^{\pm} = \pm a\ln(\Omega)\sqrt{-\frac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left(-\cot_\Theta(2ao)\pm\sqrt{pq}\csc_\Theta(2ao)\right)$$

$$\chi_{25}^{\pm} = \pm a\ln(\Omega)\sqrt{-\tfrac{3\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{2\delta}}\left(\tan_\Theta\left(\tfrac{a}{2}o\right)-\cot_\Theta\left(\tfrac{a}{2}o\right)\right)$$

where

$$o = \kappa_1 x+\kappa_2 y+\kappa_3 z+\left(2a^2\kappa_1\ln^2(\Omega)\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)\right)t$$

When $b = 0$ and $c = -a$,

$$\chi_{26}^{\pm} = \pm a\ln(\Omega)\sqrt{-\tfrac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\tanh_\Theta(ao)$$

$$\chi_{27}^{\pm} = \pm a\ln(\Omega)\sqrt{-\tfrac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\coth_\Theta(ao)$$

$$\chi_{28}^{\pm} = \pm a\ln(\Omega)\sqrt{-\frac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left(-\tanh_\Theta(2ao)\pm i\sqrt{pq}\,\mathrm{sech}_\Theta(2ao)\right)$$

$$\chi_{29}^{\pm} = \pm a\ln(\Omega)\sqrt{-\frac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left(-\coth_\Theta(2ao)\pm\sqrt{pq}\,\mathrm{csch}_\Theta(2ao)\right)$$

$$\chi_{30}^{\pm} = \pm a\ln(\Omega)\sqrt{-\tfrac{3\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{2\delta}}\left(\tanh_\Theta\left(\tfrac{a}{2}o\right)+\coth_\Theta\left(\tfrac{a}{2}o\right)\right)$$

When $a = 0$ and $b \neq 0$,

$$\chi_{31}^{\pm} = \pm\, b\ln(\Omega)\sqrt{-\frac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left[\frac{1}{2}b+\frac{p}{(\cosh_{\Theta}(bo)-\sinh_{\Theta}(bo)+p)}\right]$$

$$\chi_{32}^{\pm} = \pm\, b\ln(\Omega)\sqrt{-\frac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left[\frac{1}{2}+\frac{(\sinh_{\Theta}(vo)+\cosh_{\Theta}(bo))}{(\sinh_{\Theta}(bo)+\cosh_{\Theta}(bo)+q)}\right]$$

where

$$o = \kappa_1 x+\kappa_2 y+\kappa_3 z-\left(\frac{1}{2}b^2\kappa_1\ln^2(\Theta)\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)\right)t.$$

When $b=\lambda, c=\varpi\lambda(\varpi\neq 0)$ and $a=0$,

$$\chi_{33}^{\pm} = \pm\lambda\ln(\Theta)\sqrt{-\frac{6\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)}{\delta}}\left[\frac{1}{2}-\frac{\varpi p\Theta^{\lambda o}}{p-\varpi q\Theta^{\lambda o}}\right]$$

where

$$o = \kappa_1 x+\kappa_2 y+\kappa_3 z-\left(\frac{1}{2}\lambda^2\kappa_1\ln^2(\Theta)\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)\right)t$$

We have checked the correctness of all the solutions presented in this paper by inserting them directly into the main nonlinear equation.

3. Physical explanation

We have successfully obtained TW solutions for the mKdV-ZK equation, encompassing a range of soliton types such as dark, periodic, singular, and others. To obtain a thorough grasp of their physical behavior, we illustrated certain solutions graphically. The subsequent results were obtained and are depicted in the accompanying figures to augment our comprehension of the underlying physical phenomenon. Figs. 1 to 3 depict the 2D, 3D, and contour plots of the absolute of $\chi_i(x,y,z,t), i=1,6,33$. Fig. 1 demonstrates that the absolute values of $\chi_1(x,y,z,t)$ form a singular periodic wave solution with the duration $-5\leq t,x\leq 5$ when, while $t=0.0$ (red), $t=0.1$ (blue), $t=0.2$ (black). It can be observed from Fig. 2 that the absolute value of $\chi_6(x,y,z,t)$ exhibits a dark kink wave solution with a duration of $-5\leq t,x\leq 5$ when $a=0.5, b=1.5, c=0.75, \kappa_1=-2, \kappa_2=2, \kappa_3=0.5, \delta=-0.5, y=1, z=1, \Theta=e$, $p=1, q=1$ while $t=0.0$ (red), $t=0.1$ (blue), $t=0.2$ (black). Based on Fig. 3, it can be observed that the absolute value of $\chi_{33}(x,y,z,t)$ represents a solitary wave solution. This solution has a duration of $-5\leq t,x\leq 5$ when $\lambda=1.5, \varpi=2, \kappa_1=-2, \kappa_2=2, \kappa_3=0.5, \delta=-1.5$, $y=1, z=1, p=1, q=1.2, \Theta=e$, while $t=0.0$ (red), $t=0.1$ (blue), $t=0.2$ (black).

4. Conclusions

In this survey, a NEDA technique has been successfully applied to find TW solutions of the mKdV-ZK equation. Thereupon, some new sets of analytical solutions are determined. In the framework of the technique, the acquired solutions are determined in terms of some generalized trigonometric and hyperbolic functions where their use is very straightforward. It is worth noting that the technique employed in this article is very powerful and efficient in finding TW solutions to other well-known nonlinear models. Therefore, one of our future research goals will be to employ this effective technique in determining TW solutions for other models in the literature.

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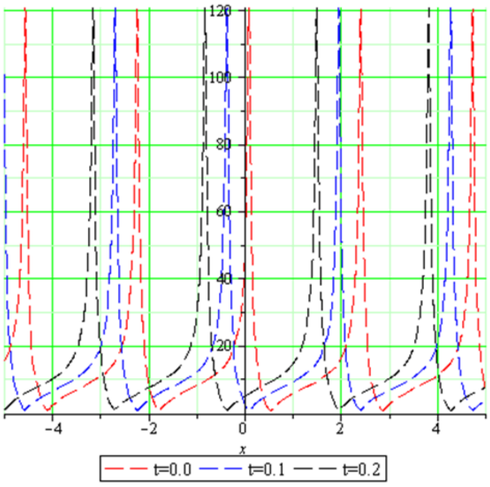
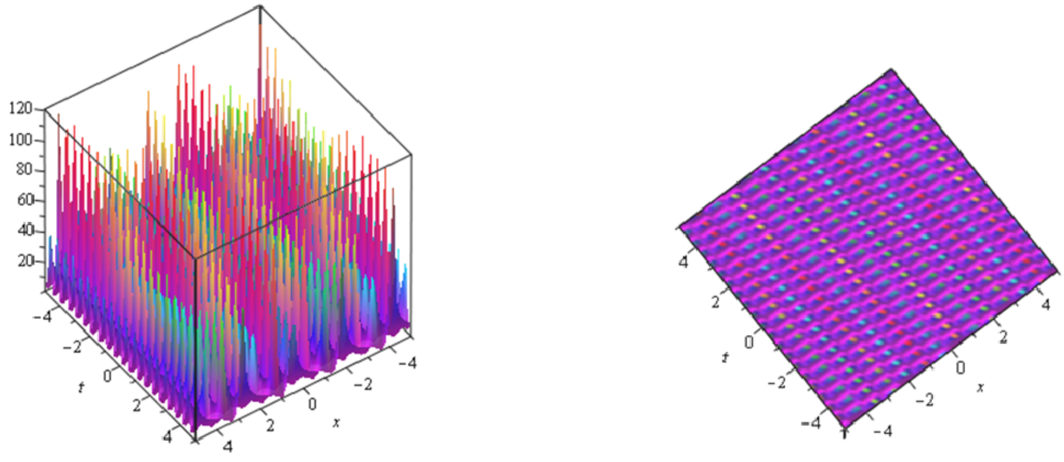


(a) 2D



(b) 3D

(c) Contour

Fig. 1. 2D, 3D, and contour plots of $|\chi_1|$ with $a = 0.5, b = 1, c = 1.5, \kappa_1 = -2, \kappa_2 = 1, \kappa_3 = 1.5, \delta = -0.5, \cdot p = 1.2, q = 1.1$ with $y = z = 1$ and $\Theta = 2.6$.

Table 1. Comparison of time performance and performance with different superpixel methods

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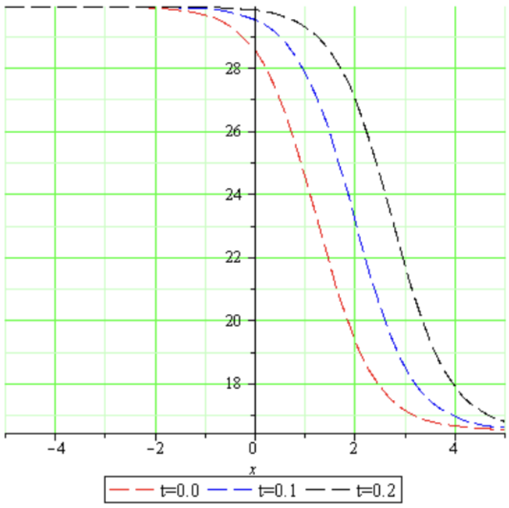
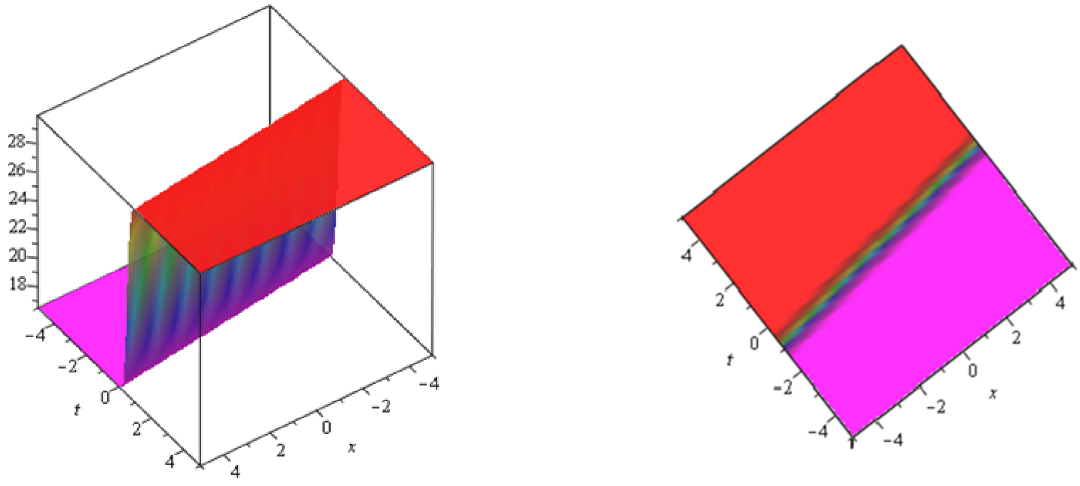


(a) 2D



(b) 3D

(c) Contour

Fig. 2. 2D, 3D, and contour plots of $|\chi_6|$ with $a = 0.5, b = 1.5, c = 0.75, \kappa_1 = -2, \kappa_2 = 2, \kappa_3 = 0.5, \delta = -0.5, \cdot p = 1, q = 1$ with $y = z = 1$ and $\Theta = e$.

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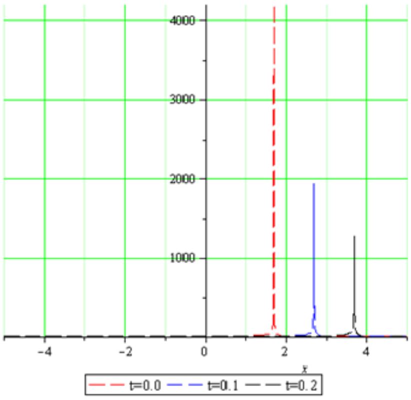
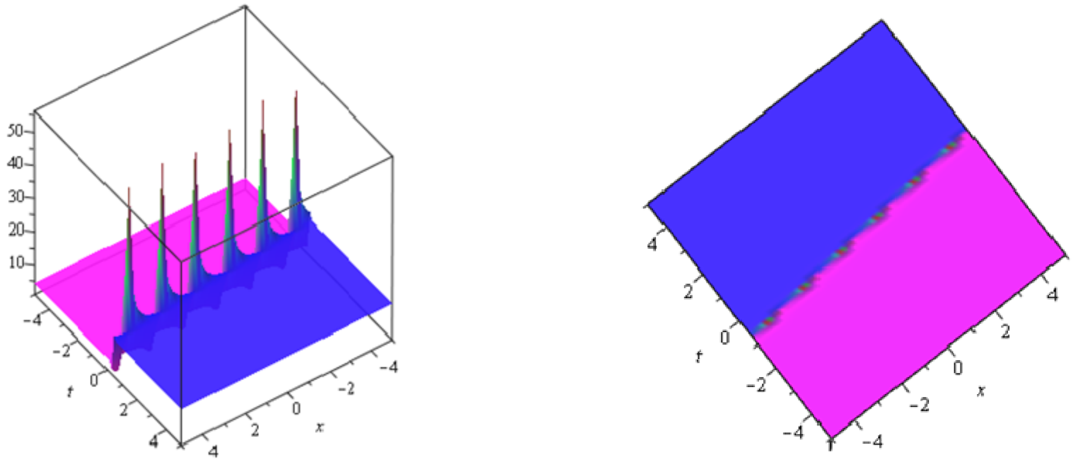


(a) 2D



(b) 3D

(c) Contour

Fig. 3. 2D, 3D, and contour plots of $|\chi_{33}|$ with $\lambda = 1.5, \varpi = 2, \kappa_1 = -2, \kappa_2 = 2, \kappa_3 = 0.5, \delta = -1.5, p = 1, q = 1.2$. with $y = z = 1$ and $\Theta = e$.

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