Forced Vibrations Of Viscoelastic Microbeam Using Fractional Order Viscoelastic Model

Zhenkun Wang and Zhaoyu Teng*

The Second XiangYa Hospital of CentralSouth University, Changsha 410000, Hunan, China

Corresponding author. E-mail: 161601007@csu.edu.cn

Received: May 02, 2023; Accepted: Sept. 03, 2023

This study investigates the dynamic behavior of a microbeam modeled using the Kelvin-Voigt fractional viscoelastic model. The microbeam is modeled using modified couple stress theory (MCST) and the fractional Kelvin-Voigt viscoelastic model. Following Hamilton's principle, the governing equations of motion were expressed as a fractional order equation with partial derivatives. Combining finite elements and finite difference methods solves the obtained equation. The finite difference is used to discretize the equations in the time domain, and finite elements and Galerkin are used to discretize the equations in the space domain. As a result of the simulations, it has been demonstrated that the derivative of the fractional order greatly influences the range and time response of the free and forced vibrations of the microbeam, as well as the effects of dampening on the microbeam. Furthermore, the effects of small size parameters and viscoelastic damping coefficient have been investigated. Based on the results, the fractional order derivative substantially influences the resonance regions.

Keywords: Microbeam; Forced vibrations; Viscoelastic; Fractional order derivative

http://dx.doi.org/10.6180/jase.202410_27(10).0010

1. Introduction

The ongoing advancements in fractional calculus, coupled with the emergence of novel theories and techniques for addressing fractional nonlinear differential equations and fractional order equations involving partial derivatives, have progressively integrated this mathematical discipline into the realm of physics [1]. Over the past decade, a substantial body of research has explored both linear and nonlinear vibrations in continuous systems characterized by viscoelastic behavior. Given that viscoelastic properties are inherent to a diverse range of materials, and these attributes can exert notable influences on a system's mechanical characteristics, the inclusion of considerations related to viscoelasticity in the analysis of microstructures has emerged as a pivotal means of achieving precision in results [2–5]. Lazopoulos et al. [6] undertook a comprehensive investigation into the distinctions among the Maxwell model, the Kelvin model, and fractional order models. Their findings revealed that, in certain instances, Maxwell's model, and occasionally Kelvin's creep model, delivered superior outcomes. In a separate study, Fatahi-Vajari et al. [7] delved into the free vibrations exhibited by a nanobeam positioned atop a viscoelastic substrate. Drawing upon non-local elasticity theory and fractional order viscoelastic theory principles, they devised a model for the nanobeam. Notably, the study leveraged the Laplace transform technique to yield an analytical solution. Moreover, Lu et al. [8] introduced a method founded on the Laplace transform to derive and assess fractional vibration models for large membranes. The results put forth in their publication align closely with those obtained through numerical methodologies.

A strong and nonlinear relationship exists between nonlinear systems' maximum oscillation amplitude and damping coefficient. Furthermore, the numerical and laboratory results can sometimes differ if the resonance frequencies are changed. Using a fractional order derivative, Guisquet et al. [9] proposed a model in which the frequency dependence of the damping is adjusted to match the material's behavior. Based on laboratory results, the presented model accurately predicts increased damping with increasing vibration amplitude. The nonlinear vibration behavior of a viscoelastic sheet with a fractional order was studied by Amabili et al. [10]. They used the nonlinear finite element method to solve the structural equation based on fractional order viscoelasticity. The article also examines the behavior of beams with different boundary conditions and loads. Askarian et al. [11] examined the nonlinear vibration behavior of a viscoelastic tube containing fluid in a fractional-order model. Based on beam theory and Hamilton's principle, the fractional model and nonlinear geometry were considered in the development of the tube model. According to the results of this research, the fractional order model predicts greater values for the fluid-conveying viscoelastic pipe amplitude than previous models. According to the Kelvin-Voigt fractional order viscoelastic model, Javadi et al. [12] studied the nonlinear vibrations of beams. Javadi et al. [13] studied the forced nonlinear dynamics of a viscoelastic pipe by a fractional order fluid model and the influence of the fractional order equation on the primary and secondary resonances. To model the problem, Euler-Bernoulli beam theory and Von-Karman nonlinear theory were used. After discretization by the Galerkin method, the multiple scales method is used in this reference to solve the problem. The dynamics of composite beams containing viscoelastic layers have been studied by Lewandowski et al. [14]. The Zener fractional viscoelastic model was used to model the viscoelastic behavior. Carbon nanotubes' free and forced vibrations have been examined in reference [15]. This reference considers viscoelastic behavior as a fractional model. The Euler-Bernoulli theory and van Karman theory have been used to model the nanotube. Loghman et al. [16] delved into analyzing nonlinear vibrations in a fractional viscoelastic functionally graded micro-beam, employing fractional calculus as a critical mathematical tool. The Modified Couple Stress Theory (MCST) is applied to capture micro-scale effects. The micro-beam is characterized using the Euler-Bernoulli theory and nonlinear strain relations following Von Karman's approach. Furthermore, the viscoelastic properties of the micro-beam are incorporated by using the fractional Kelvin-Voigt viscoelastic model. Qiu et al. [17] introduced a novel non-classical Euler–Bernoulli beam mod, which accounts for the size-dependent effects typically neglected in classical continuum mechanics. This new model, called the fractional viscoelastic nanobeam model, is formulated based on the fractional Kelvin–Voigt viscoelastic model and is established using Hamilton's principle. It comprehensively investigates the combined impacts of non-local elasticity, modified couple stress, and surface energy, providing a more accurate representation of nanobeam behavior. Bayat et al. [18] introduced the Homotopy Perturbation Method (HPM) as a robust analytical tool. It focuses on the analytical analysis of the free vibration of an electrostatically actuated microbeam. The study includes an examination of the system's response concerning critical parameters. Comparative analyses are also conducted to validate the findings against results from other researchers and numerical solutions. In another study, Bayat et al. [19] employed He's variational approach method to address the problem of large amplitude free vibration. Specifically, we focus on a grounded mass connected to a series of linear and nonlinear springs. The oscillatory system in question is characterized by a nonlinear ordinary differential equation comprising linear and nonlinear stiffness elements. Through straightforward mathematical manipulations within this method, we derive the system's natural frequencies.

This study investigated the vibrations of the viscoelastic microbeams by employing the fractional Kelvin-Voigt model. In this study, the microbeam is modeled using linear strains, MCST, and the fractional Kelvin-Voigt viscoelastic model. Applying Hamilton's principle, the motion equations are obtained as a fractional order equation with partial derivatives. The obtained equation has been solved using a combination of finite element and finite difference methods for the first time in this research. A study of the influence of various parameters on the dynamic behavior of these beams has been conducted following the extraction of the results.

The main contributions of the paper can be summarized as follows:

(i) The paper introduces a novel modeling approach for studying the dynamic behavior of microbeams with viscoelastic properties. It combines modified couple stress theory (MCST) and the fractional Kelvin-Voigt viscoelastic model, providing a comprehensive framework for characterizing the microbeam's behavior.

(ii) By applying Hamilton's principle, the authors derive the equations of motion for the microbeam as fractional-order equations with partial derivatives. The mentioned equations are a significant contribution as they allow for incorporating fractional calculus principles into the analysis of viscoelastic microstructures.

(iii) The paper presents a numerical solution method that combines finite element and finite difference techniques. This innovative approach enables the efficient solution of the fractional order equations, considering both time and space domains.

(iv) The study systematically investigates the influence of various parameters on the dynamic behavior of the microbeam. It specifically examines the impact of the fractional order derivative, small-size parameters, and viscoelastic damping coefficients. This analysis provides valuable insights into how these parameters affect the microbeam's response.

(v) The research highlights a key finding that the choice of the fractional order derivative significantly influences the resonance regions of the microbeam. This result underscores the importance of selecting an appropriate fractional order to capture the system's dynamic behavior accurately.

In summary, the paper's main contributions lie in developing a novel modeling framework for viscoelastic microbeams, formulating fractional order equations, introducing a numerical solution method, and systematically analyzing parameter influences on the microbeam's dynamic response. These contributions enhance our understanding of microstructures with viscoelastic behavior and their applications in various engineering fields.

2. Motion equations

This section uses Hamilton's principle method to determine Euler-Bernoulli equations of motion for microbeams with fractional-order viscoelastic models and simple supports. As shown in Figure 1, a microbeam has a length of L, a modulus of elasticity E, a density of ρ, and a cross-section area of A. Euler-Bernoulli beam theory states that the non-zero components of strain are as follows:

$$\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2} \tag{1}$$

in which ε_x stands the axial strain, and w denotes the transverse displacement of the microbeam.

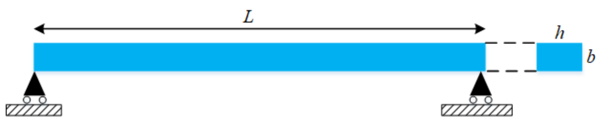

Fig. 1. Viscoelastic microbeam geometry.

The kinetic energy of the microbeam is obtained in the form of:

$$T = \frac{1}{2}\int_0^L \iint_A \rho\left(\frac{\partial w}{\partial t}\right)^2 dAdx = \frac{1}{2}\rho A\int_0^L\left(\frac{\partial w}{\partial t}\right)^2 dx \tag{2}$$

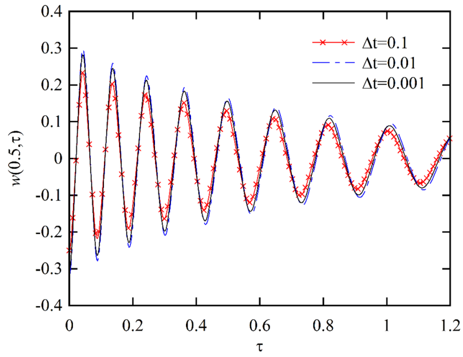

Fig. 2. Convergence investigation of the finite difference method for different time steps.

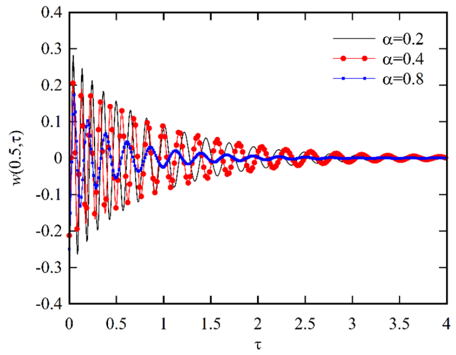

Fig. 3. The effect of the order of the fractional derivative on the free vibration response of the microbeam.

Also, the potential energy (U) is obtained according to the MCST in the form of the following relationship:

$$U = \frac{1}{2}\int_0^L \left[\sigma_x\varepsilon_x + m_{xy}\chi_{xy} + m_{yx}\chi_{yx}\right] dx \tag{3}$$

where m_{ij} is the micro stress tensor, and χ_{ii} is the symmetric rotation gradient tensor, which is calculated as the following equation:

$$\begin{aligned} m_{xy} &= m_{xy} = \mu l^2\frac{\partial^2 w}{\partial x^2}, \\ \chi_{xy} &= \chi_{xy} = -\frac{1}{2}\frac{\partial^2 w}{\partial x^2}, \\ \sigma_x &= E\varepsilon_x \end{aligned} \tag{4}$$

where l is the dimension component of the beam. It is assumed that the microbeam is subjected to an external force and is located on a viscous foundation. In this way, the work done by outside forces is obtained as follows:

$$\delta W_{nc} = \int_0^L (F - C_d\dot{w})\,\delta w dx \tag{5}$$

where F denotes the external force applied to the microbeam, and C_d is the viscoelastic damping of the foundation.

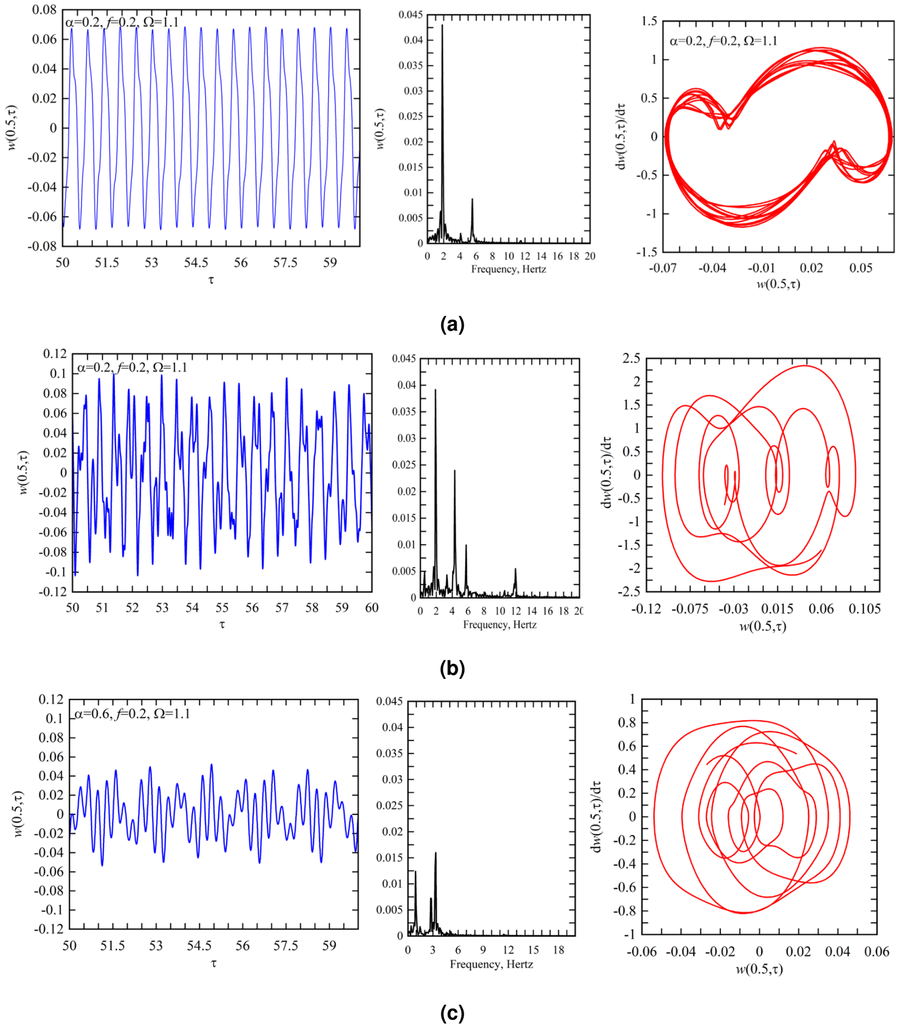

Fig. 4. The effect of the fractional order derivative on the steady-state response of the forced vibrations of the microbeam.

According to Eqs. (2)-(5) and using Hamilton's principle, the governing vibration equation of motion of the microbeam located on the viscoelastic foundation is obtained as follows:

$$\rho A \frac{\partial^2 w}{\partial t^2} + C_d \frac{\partial w}{\partial t} + \left(EI + \mu A l^2\right) \frac{\partial^4 w}{\partial x^4} = F(x)\cos(\omega t) \quad (6)$$

where ω is the frequency of the excitation force.

Considering that the simply supported boundary conditions, it can be defined that:

$$w(0,t) = w''(0,t) = 0, \quad w(L,t) = w''(L,t) = 0 \quad (7)$$

The fractional order viscoelastic Kelvin-Voigt model is written as the following equation [20]:

$$\sigma = E\left(\varepsilon + \bar{g}\frac{\partial^\alpha}{\partial t^\alpha}\varepsilon\right) \quad (8)$$

where α is the derivative order that can change from 0 to 1 and $\bar{g}$ is the viscoelasticity coefficient.

Using Eqs. (6) and (8), the equation of motion governing the viscoelastic microbeam is obtained as follows:

$$\rho A \frac{\partial^2 w}{\partial t^2} + C_d \frac{\partial w}{\partial t} + \left(EI + \mu A l^2\right) \frac{\partial^4 w}{\partial x^4} + \left(EI + \mu A l^2\right) \frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^4 w}{\partial x^4}\right) = F(x)\cos(\omega t) \quad (9)$$

Using the following dimensionless variables

$$\bar{w} = \frac{w}{l}, \bar{x} = \frac{x}{l}, \tau = t\sqrt{\frac{EI}{\rho A L^4}},$$
$$c = \frac{c_d L}{EI}\sqrt{\frac{EI}{\rho A L^4}}, f = \frac{FL^4}{EIh}, \Omega = \omega\sqrt{\frac{\rho A L^4}{EI}},$$
$$\gamma = \bar{g}\left(\frac{EI}{\rho A L^4}\right)^\alpha, \eta_1 = \frac{\mu A l^2}{EI},$$
$$\eta_2 = \frac{\mu A l^2}{EI}\bar{g}\left(\frac{EI}{\rho A L^4}\right)^\alpha \quad (10)$$

the motion equation can be written in the following dimensionless form:

$$\frac{\partial^2 \bar{w}}{\partial \tau^2} + c_d \frac{\partial \bar{w}}{\partial \tau} + (1+\eta_1)\frac{\partial^4 w}{\partial x^4} + (\gamma+\eta_2)\frac{\partial^\alpha}{\partial \tau^\alpha}\left(\frac{\partial^4 \bar{w}}{\partial \bar{x}^4}\right) = f(\bar{x})\cos(\Omega\tau) \tag{11}$$

In the following, superscripts are omitted to simplify.

3. Solving the motion equation

Eq. (11) is a linear partial differential equation of fractional order. To discretize the Caputo derivative, the following relation is used [21]:

$$D^\alpha u^n = \frac{1}{\Delta t^\alpha}\left[b_0 u^n - \sum_{m=0}^{n-1}(b_{n-m-1}-b_{n-m})\, u^m - b_n u^0\right] \tag{12}$$

$$b_m^\alpha = \frac{1}{\Gamma(2-\alpha)}\left[(m+1)^{1-\alpha} - m^{1-\alpha}\right] \tag{13}$$

in which Γ denotes the symbol of the gamma function, n is the step number, u is a symbolic parameter, and D is the Caputo derivative symbol. To discretize differential Eq. (11), the following relations are used:

$$\frac{\partial^2 w}{\partial x^2} = \frac{w^{n+1} - 2w^n + w^{n-1}}{\Delta t^2}$$
$$\frac{dw}{d\tau} = \frac{w^{n+1} - w^{n-1}}{2\Delta t}$$
$$D^\alpha[w_{,xxxx}] = \frac{1}{\Delta t^\alpha}\left[b_0 w_{xxx}^{n+1} - \sum_{m=0}^{n}(b_{n-m}+b_{n-m+1})\, w_{xxx}^m - b_{n+1} w_{xxx}^0\right] \tag{14}$$

Using eq. (14), differential Eq. (11) is discretized as the following relation:

$$\frac{w^{n+1} - 2w^n + w^{n-1}}{\Delta t^2} + c_d\left(\frac{w^{n+1} - w^{n-1}}{2\Delta t}\right) + (1+\eta_1)V_{xx}^{n+1} + \frac{(\gamma+\eta_2)\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\left[b_0 V_{xx}^{n+1} - \sum_{m=0}^{n}(b_{n-m}+b_{n-m+1})V_{xx}^m - b_{n+1}V_{xx}^0\right] = f(\bar{x})\cos(\Omega\tau) \tag{15}$$
$$V^{n+1} = w_{xx}^{n+1}$$

In the following, the finite element method is applied for discretization in the space domain in the form of the following relation:

$$w^n(x) = \sum_{k=1}^{N}\lambda_k^n \phi_k(x), \quad V^n(x) = \sum_{k=1}^{N}\theta_k^n \phi_k(x) \tag{16}$$

where λ_k^n and θ_k^n are the time vectors created in the finite difference method. Also, N stands for the number of functions used in the Galerkin method. Using the Galerkin method and selecting functions $\phi_k(x)$ in the form of the following relationship, the problem can be solved:

$$\phi_k(x) = \sin(k\pi x) \tag{17}$$

4. Numerical results

Epoxy microbeam is selected with the numerical values found in reference [22]:

$E = 1.44\text{GPa}, l = 1.7\mu\text{m}, h = 5.4L, b = 2h, L = 200l, \mu = 521.7\text{MPa}, \rho = 1220\ \text{kg}\cdot\text{m}^{-3}$

The convergence of the finite difference method is illustrated in Figure 2. According to Figure 2, the finite difference method is convergent. In the following, it is considered for obtaining the results.

Figure 3 illustrates the effect of the fractional derivative's order on the microbeam's free vibration response. Based on Figure 3, the fractional derivative significantly impacts the system's damping, with its increase resulting in a decrease in vibration amplitude and a faster damping rate. As shown in Figure 4, the fractional order derivative significantly influences the steady-state response of the microbeam when subjected to forced vibrations. According to Figure 4, the order of the fractional derivative can substantially affect the force vibrations' amplitude.

The effect of the coefficient $\bar{g}$ is visible in Figure 5. As shown in Figure 5, the viscoelastic damping coefficient can contribute significantly to faster damping of vibrations. In addition to having an effective effect on vibrations with the normal viscoelastic model, this factor is also effective in vibrations of fractional model systems.

An illustration of the influence of external force amplitude on the steady-state response of forced vibrations is displayed in Figure 6. Based on Figure 6, it can be concluded that the force coefficient plays a significant role in the vibrations of the microbeam under the fractional order model. To examine the effects of the small size parameter, Figure 7 shows the system response for different values of the small size parameter. As demonstrated in Figure 7, small-size parameters significantly impact vibration. The oscillation time and amplitude have increased as the small size parameter increases. Because the stiffness of the beam decreases with increasing length and decreasing thickness, the natural frequency also decreases, and the oscillation amplitude and time increase.

Based on Figure 8, it can be seen that the excitation frequency can have a variety of effects on the vibration amplitude. As can be seen from the figure, the influence

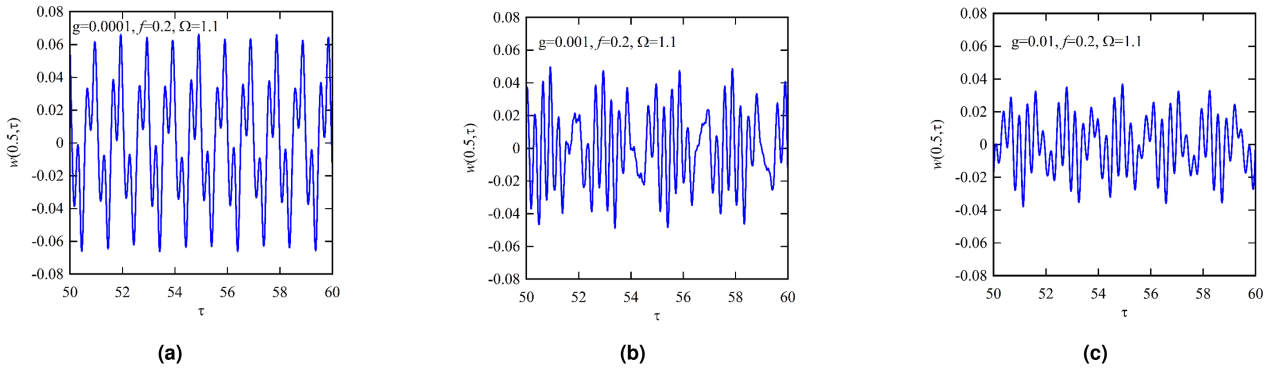

Fig. 5. Influence of damping on the steady-state response of forced vibrations, (a) $\bar{g}$ = 0.0001, (b) $\bar{g}$ = 0.001 and (c) = 0.01.

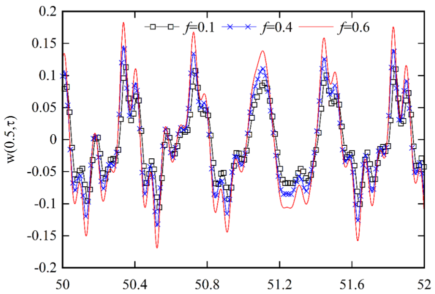

Fig. 6. Effect of force amplitude on steady-state response in forced vibrations.

of fractional derivative order is much more pronounced when the excitation frequency is near the resonance frequency. As a result, where the excitation frequency is far from the resonance mode, the derivative order does not significantly impact the response. In contrast, if the excitation frequency is near the resonance frequency, the derivative order strongly affects the amplitude of the fluctuations.

Illustrated in Fig. 9 is the influence of initial amplitude on the dynamic time trace of the mid-point of a fractional viscoelastic microbeam associated. In agreement with this figure, due to viscous influences, the highest amplitude deteriorating with time. Also, for bigger values of initial amplitude, the acceleration of decay is greater. This is due to the actuality that the frequency of vibration reduces with decreasing the dynamic amplitude.

The dependence of the dimensionless damped natural frequency (Ω/ω_1) on the order of the derivative order for the 1st vibration mode is presented in Fig. 10. It can be seen that the value of the first damped dimensionless natural frequency virtually does not dependent on the order of the fractional derivative.

By ignoring the effect of small sizes, the governing equation is obtained in the form of the relation presented in reference [23]. In this case, the comparison of the results of the current research with Ref. [23] is shown in Fig. 11. As is clear, the results of the present study are in good agreement with the theoretical results presented by Permoon et al. [23].

5. Conclusion

The Kelvin-Voigt fractional order model allows for more accurate and flexible damping models in viscoelastic materials. An investigation of the vibration behavior of viscoelastic microbeams was performed by considering fractional viscoelasticity using Galerkin and finite difference techniques. Based on the results of this article, it is possible to model viscoelastic materials more realistically. It was also demonstrated that the influence of fractional order derivative in the resonance mode is much stronger than in the off-resonance mode. The presented results make it possible to understand the significant impact of the fractional order derivative on damping. The vibrations occur faster, and their amplitude decreases drastically when the derivative order of fractional damping is increased. Both free and forced vibrations are affected by this. In addition to the viscoelastic damping coefficient, the small size parameter plays an important role in damping, and changing these parameters alters the results.

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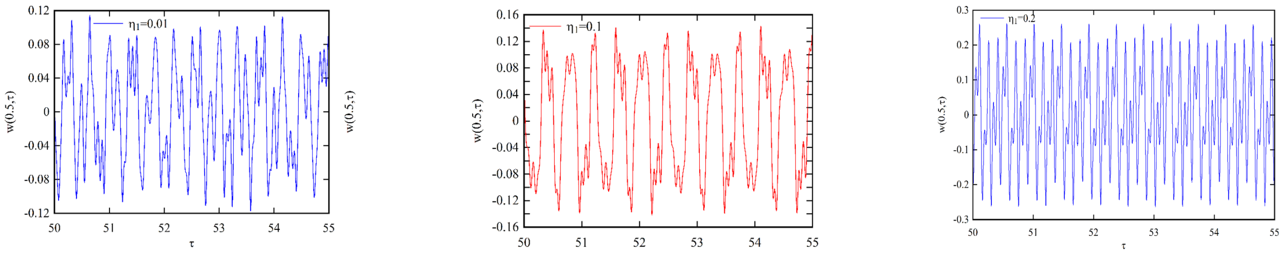

(a) (b) (c)

Fig. 7. Effect of small size parameter on steady-state response in forced vibrations, (a) $\eta_1 = 0.01$, (b) $\eta_1 = 0.1$ and (c) $\eta_1 = 0.2$.

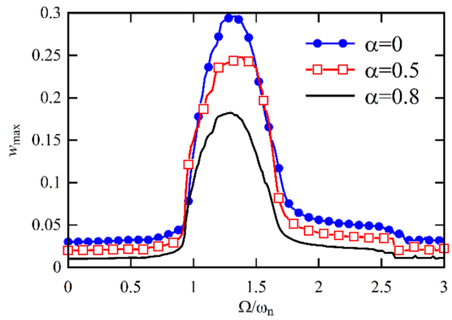

Fig. 8. Effect of the external excitation force frequency on the maximum amplitude of microbeam oscillations for different values of the fractional derivative.

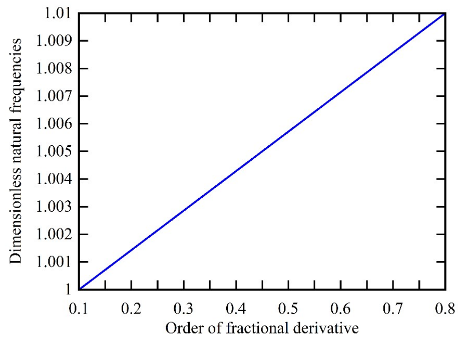

Fig. 10. Dimensionless natural frequencies versus order of fractional derivative.

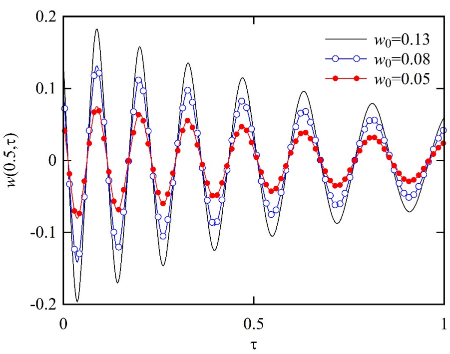

Fig. 9. Influence of initial dynamic amplitude on the microbeam oscillations.

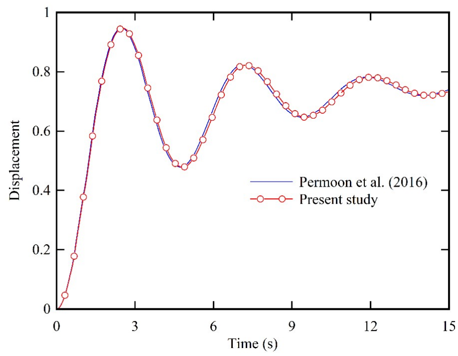

Fig. 11. Comparison of the presented results with the results by Permoon et al. [23].

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