Interpolative fuzzy Lipschitz summing maps

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Building upon the interpolative classical results of D. Achour, P. Rueda, and R. Yahi for Lipschitz ideals we define the interpolative fuzzy Lipschitz ideal concept for fuzzy Lipschitz operators between fuzzy metric spaces and complete fuzzy normed spaces which is a natural generalization of the notion of absolutely (crisp) Lipschitz (p, θ)-summing maps. The fuzzy Lipschitz norm of the aforementioned notion is defined and prove its fuzzy Lipschitz norm is a fuzzy real number. Afterwards we establish a fundamental characterizations of absolutely interpolative fuzzy Lipschitz (p, θ)-summing map. This is over by establishing a fuzzy version of nonlinear Pietsch Domination Theorem. Finally, we raise some open problems which we think are interesting.

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1. Notations and preliminaries

The symbols $\mathbb{R}^+, \mathbb{R}$, and $\mathbb{N}$ stand for the sets of all positive real numbers, of all real numbers and of all positive integers, respectively. The order pair $(F, \|\cdot\|)$ stands for Banach space. The order pairs $(X, \mathbb{D}_X, x_0)$ and $(Y, \mathbb{D}_Y, y_0)$ will denote pointed metric spaces. A map S from $(X, \mathbb{D}_X, x_0)$ into $(Y, \mathbb{D}_Y, y_0)$ is called Lipschitz if there is a nonnegative constant $\mathfrak{O}$ such that $\mathbb{D}_Y(Sx, Sy) \leq \omega \mathbb{D}_X(x, y)$ for all x and y in X. The smallest possible C is the Lipschitz constant of S denoted by $\mathrm{Lip}(S)$. The class of all Lipschitz maps from $(X, \mathbb{D}_X, x_0)$ into $(Y, \mathbb{D}_Y, y_0)$ is denoted by $\mathcal{L}(X, Y)$. The Banach space of real-valued Lipschitz functions defined on $(X, \mathbb{D}_X, x_0)$ that send the special point x_0 into 0 with the Lipschitz norm $\mathrm{Lip}(\cdot)$ will be denoted by $X^\#$.

Definition 1.1. [1] Let ξ and δ belong to $\aleph$. Define a partial ordering by $\xi \preceq \delta$ if and only if $\xi_\alpha^- \leq \delta_\alpha^-$ and $\xi_\alpha^+ \leq \delta_\alpha^+$ for all $\alpha \in (0, 1]$, where $\aleph$ is a fuzzy real number.

Lemma 1.1. [2] Let ξ and δ belong to $\aleph$. and let $[\xi]_\alpha = [\xi_\alpha^-, \xi_\alpha^+]$, $[\delta]_\alpha = [\delta_\alpha^-, \delta_\alpha^+]$. Then

$$[\xi \oplus \delta]_\alpha = [\xi_\alpha^- + \delta_\alpha^-, \xi_\alpha^+ + \delta_\alpha^+],$$
$$[\xi \ominus \delta]_\alpha = [\xi_\alpha^- - \delta_\alpha^+, \xi_\alpha^+ - \delta_\alpha^-],$$
$$[\xi \odot \delta]_\alpha = [\xi_\alpha^- \cdot \delta_\alpha^-, \xi_\alpha^+ \cdot \delta_\alpha^+], \text{for } \xi, \delta \in \aleph^+.$$

Definition 1.2. [2] Let X be a non-empty set and $\overline{\mathbb{D}}_X$ be a function from $X \times X$ into $\aleph^+$. An ordered pair $(X, \bar{D}_X)$ is said to be a fuzzy metric space with a fuzzy norm $\overline{\mathbb{D}}_X$ on $X \times X$ if the following conditions are satisfied:

1. $\overline{\mathbb{D}}_X(x, y) = \tilde{0}$ if and only if $x = y$.
2. $\overline{\mathbb{D}}_X(x, y) = \overline{\mathbb{D}}_X(y, x), \forall x, y \in X$.
3. $\overline{\mathbb{D}}_X(x, y) \preceq \overline{\mathbb{D}}_X(x, z) \oplus \overline{\mathbb{D}}_X(z, y), \forall x, y$, and $z \in X$.

2. Introduction

We start by recalling the definition of Lipschitz ideal concept of [3] as follows. Suppose that, for every pair of metric spaces X and Y, we are given a subset $\mathcal{W}^L(X, Y)$ of $\mathcal{L}(X, Y)$. The class

$$\mathcal{W}^L := \bigcup_{X,Y} \mathcal{W}^L(X, Y)$$

is said to be a Lipschitz ideal, if the following conditions are satisfied:

($\mathbf{LI}_0$) If $Y = F$, then $g \odot e \in \mathcal{W}^L(X,F)$ for $g \in X^{\#}$ and $e \in F$.

($\mathbf{LI}_1$) $BTA \in \mathcal{W}^L(X_0, Y_0)$ for $A \in \mathscr{L}(X_0, X)$, $T \in \mathcal{W}^L(X,Y)$, and $B \in \mathscr{L}(Y, Y_0)$, where X_0 and Y_0 be metric spaces. Condition ($\mathbf{LI}_0$) implies that $\mathcal{W}^L$ contains nonzero Lipschitz operators.

One important example of Lipschitz ideal is the class of Lipschitz p-summing maps defined by J.Farmer and W. Johnson in [4] as follows. A Lipschitz map S from $(X, \mathbb{D}_X, x_0)$ into $(Y, \mathbb{D}_Y, y_0)$ is called Lipschitz p-summing $(1 \le p < \infty)$ if and only if there is a constant ≥ 0 such that

$$\left[\sum_{j=1}^{m} \mathbb{D}_Y\left(Sx_j, Sy_j\right)^p\right]^{\frac{1}{p}} \le \omega \cdot \sup_{f \in B_{X^{\#}}} \left[\sum_{j=1}^{m} \left|fx_j - fy_j\right|^p\right]^{\frac{1}{p}}$$

for arbitrary sequences $\left(x_j\right)_{j=1}^{m}, \left(y_j\right)_{j=1}^{m}$ in X, and $m \in \mathbb{N}$. The Lipschitz p-summing norm $\mathbf{P}_p^L(T)$ is then the smallest possible constant C. They proved the fundamental theorem of Lipschitz p-summing maps called Lipschitz Pietsch Domination Theorem as follows.

Theorem 2.1. [4] Let $1 \le p < \infty$. For every Lipschitz map S from $(X, \mathbb{D}_X, x_0)$ into $(Y, \mathbb{D}_Y, y_0)$ and $\omega \ge 0$, the following are equivalent:

1. $P_p^L(S) \le \omega$.
2. There is a probability measure $\varkappa$ on B_X such that

$$\mathbb{D}_X(Sx, Sy)^p \le {}^p \cdot \int_{B_{X^{\#}}} |f(x) - f(y)|^p d\varkappa(f).$$

For more information on the (crisp) Lipschitz ideals can be found in the monographs [3, 5–9].

We now describe the contents of the paper. In Section 3, One particularly important class of interpolative fuzzy Lipschitz ideals is the class of absolutely fuzzy Lipschitz (p,θ)-summing maps between arbitrary fuzzy pointed metric spaces $(1 < p < \infty, 0 < \theta \leqq 1)$ which is an innate extension of absolutely (crisp) Lipschitz (p,θ)-summing maps between arbitrary metric spaces defined. We define the absolutely fuzzy Lipschitz (p,θ)-summing norm $\mathscr{F}\mathrm{P}_{p,\theta}^L(T)^\sim$ and show that it is a fuzzy real number. We prove some properties and inclusion results of the aforementioned notion and establish that the resulting class of fuzzy Lipschitz maps is fuzzy Lipschitz ideal. We prove the fundamental characterization of absolutely fuzzy Lipschitz (p,θ)-summing maps called fuzzy Lipschitz version of Pietsch Domination Theorem. In Section 4, we raise some open problems which we think are interesting.

3. Fuzzy lipschitz ideals between fuzzy pointed metric spaces

Before introducing the nonlinear theory of fuzzy Lipschitz ideals between arbitrary fuzzy pointed metric spaces the reader can be aware of the linear theory of fuzzy operator ideals between arbitrary fuzzy normed spaces in [10]. Now we construct the terminology of fuzzy Lipschitz ideals between fuzzy pointed metric spaces and fuzzy normed spaces as follows.

Suppose that, for every fuzzy pointed metric space X and fuzzy normed space F, we are given a subset $\mathscr{F}\mathcal{W}^L(X,F)$ of $\mathscr{F}Lip(X,F)$. The class

$$\mathscr{F}\mathcal{W}^L := \bigcup_{X,F} \mathscr{F}\mathcal{W}^L(X,F)$$

is said to be a fuzzy Lipschitz ideal if the following conditions are satisfied:

($\mathbf{I_0}$) If $Y = F$, then $f \circledcirc e \in \mathscr{F}\mathcal{W}^L(X,F)$ with $\mathcal{W}^L(f \circledcirc e)^\sim \preceq \|e\|_\sim$ for $f \in \mathfrak{B}_{\mathscr{F}_{X^{\#}}}$ and $e \in F$.

($\mathbf{I_1}$) $S + T \in \mathscr{F}\mathcal{W}^L(X,F)$ with $\mathcal{W}^L(S+T)^\sim \preceq \mathcal{W}^L(S)^\sim \oplus \mathcal{W}^L(T)^\sim$ for S and $T \in \mathscr{F}\mathcal{W}^L(X,F)$.

($\mathbf{I_2}$) $AT \in \mathscr{F}\mathcal{W}^L(X,F)$ with $\mathcal{W}^L(AT)^\sim \preceq \mathscr{F}Lip(A)^\sim \odot \mathcal{W}^L(T)^\sim$ for $T \in \mathscr{F}\mathcal{W}^L(X,G)$ and $A \in \mathscr{F}Lip(G,F)$, where $\mathcal{W}^L$ is a function from $\mathscr{F}\mathcal{W}^L$ into $\aleph^+$. The condition of ($\mathbf{I_0}$) implies that $\mathscr{F}\mathcal{W}^L$ contains nonzero fuzzy Lipschitz operators.

3.1. Interpolative fuzzy Lipschitz ideals

Definition 3.2. Let $0 \le \theta < 1$ and let $\mathscr{F}\mathcal{W}^L$ be a fuzzy Lipschitz ideal. A fuzzy Lipschitz operator T from X into F belongs to $\mathscr{F}\mathcal{W}_\theta^L(X,F)$ if there exist fuzzy real number $\xi \in \aleph^+$, complete fuzzy normed space G and fuzzy Lipschitz operator $S \in \mathscr{F}\mathcal{W}^L(X,G)$ such that

$$\|Tx - Ty\| \preceq \xi^\theta \odot \|Sx - Sy\|^{1-\theta} \odot \overline{\mathbb{D}}_X(x,y)^\theta, \forall x,y \in X \tag{1}$$

Proposition 3.1. *If $0 \le \theta < 1$, then $\mathscr{F}\mathcal{W}_\theta^L$ be a fuzzy Lipschitz ideal.*

Proof First to show that the algebraic condition of ($\mathbf{I}_0$). Let $x, y \in X, e \in F$ and $g \in \mathfrak{B}_{\mathscr{F}X^{\#}}$ we get

$$\begin{aligned} &\|g \odot e(x) - g \odot e(y)\| \preceq \|e\|^\theta \\ &\odot \|g \odot e(x) - g \odot e(y)\|^{1-\theta} \odot \overline{\mathbb{D}}_X(x,y)^\theta \end{aligned} \tag{2}$$

Since the fuzzy Lipschitz operator $S := g \odot e \in \mathscr{F}\mathcal{W}^L(X,F)$ and $\xi := \|e\| \in \aleph^+$, we have $g \odot e \in \mathscr{F}\mathcal{W}_\theta^L(X,F)$. To prove the algebraic condition of ($\mathbf{I_1}$). Let

T_1 and T_2 be in $\mathscr{FW}_\theta^L(X,F)$. There are fuzzy real numbers $\xi_i \in \aleph^+$, complete fuzzy normed spaces G_i and fuzzy Lipschitz operators $S_i \in \mathscr{FW}^L(X,G_i), i = 1,2$ such that

$$\begin{aligned} &\|T_i x - T_i y\| \preceq \xi_i^\theta \\ &\odot \|S_i x - S_i y\|^{1-\theta} \odot \overline{\mathbb{D}}_X(x,y)^\theta, \forall x,y \in X \end{aligned} \tag{3}$$

Put $G := G_1 \boxplus_1 G_2$ and the fuzzy Lipschitz operator $S := J_1 S_1 + J_2 S_2 \in \mathscr{FW}^L(X,G)$, we get for all x,y in X

$$\begin{aligned} &\|(T_1+T_2)x - (T_1+T_2)y\| \\ &\preceq \|T_1 x - T_1 y\| \oplus \|T_2 x - T_2 y\| \\ &\preceq \sum_{i=1}^{2} \xi_i^\theta \odot \|S_i x - S_i y\|^{1-\theta} \odot \overline{\mathbb{D}}_X(x,y)^\theta \\ &\preceq \left(\sum_{i=1}^{2} \|S_i x - S_i y\|\right)^{1-\theta} \odot \left(\sum_{i=1}^{2} \xi_i\right)^\theta \odot \overline{\mathbb{D}}_X(x,y)^\theta \\ &= (\xi_1 \oplus \xi_2)^\theta \odot \|Sx - Sy\|_1^{1-\theta} \odot \overline{\mathbb{D}}_X(x,y)^\theta. \end{aligned} \tag{4}$$

Since the fuzzy Lipschitz operator $S \in \mathscr{FW}^L(X,\tilde{G})$ and $\xi := \xi_1 \oplus \xi_2 \in \aleph^+$, we obtain $T_1 + T_2 \in \mathscr{FW}_\theta^L(X,F)$. To show the algebraic condition of $(\mathbf{I_2})$. Let $T \in \mathscr{FW}_\theta^L(X,F)$ and $B(F,W)$. Then

$$\begin{aligned} \|BTx - BTy\| &\preceq \|B\| \odot \|Tx - Ty\| \\ &\preceq \|B\| \odot \xi^\theta \odot \|Sx - Sy\|^{1-\theta} \odot \overline{\mathbb{D}}_X(x,y)^\theta \\ &= \left(\|B\|^{\frac{1}{\theta}} \odot \xi\right)^\theta \odot \|Sx - Sy\|^{1-\theta} \odot \overline{\mathbb{D}}_X(x,y)^\theta \end{aligned}$$

Since $\gamma := \|B\|^{\frac{1}{\theta}} \odot \xi \in \aleph^+$and a fuzzy Lipschitz operator $S \in \mathscr{FW}^L(X,G)$, we obtain $BT \in \mathscr{FW}_\theta^L(X,W)$.

The preceding proposition give reasons for the following open question.

Conjecture 1. Under which fuzzy Lipschitz norm the interpolative fuzzy Lipschitz operator ideal $\mathscr{FW}_\theta^L$ be a complete fuzzy normed space?

Proposition 3.2. *Let* $0 \le \theta, \theta_1, \theta_2 < 1$. *Then the following holds.*

(a) If $\theta_1 \le \theta_2$ and $\theta_2 \ne 0$, then $\mathscr{FW}_{\theta_1}^L \subset \mathscr{FW}_{\theta_2}^L$.

(b) $\left(\mathscr{FW}_{\theta_1}^L\right)^{\theta_2} \subset \mathscr{FW}_{\theta_1+\theta_2-\theta_1\cdot\theta_2}^L$.

Proof To verify (a), let $T \in \mathscr{FW}_{\theta_1}^L(X,F)$ and $\epsilon > 0$. Then

$$\|Tx - Ty\| \preceq \xi^{\theta_1} \odot \|Sx - Sy\|^{1-\theta_1} \odot \overline{\mathbb{D}}_X(x,y)^{\theta_1}, \forall x,y \in X,$$

holds for a suitable fuzzy real number $\xi \in \aleph^+$, complete fuzzy normed space G, and fuzzy Lipschitz operator $S \in \mathscr{FW}^L(X,G)$. From our hypothesis we obtain

$$\begin{aligned} &\|Tx - Ty\| \preceq \left(\xi^{\frac{\theta_1}{\theta_2}} \odot \|S\|^{\frac{\theta_2-\theta_1}{\theta_2}}\right)^{\theta_2} \odot \|Sx - Sy\|^{1-\theta_2} \\ &\odot \overline{\mathbb{D}}_X(x,y)^{\theta_2}, \forall x,y \in X. \end{aligned}$$

Since $\gamma := \xi^{\frac{\theta_1}{\theta_2}} \odot \|S\|^{\frac{\theta_2-\theta_1}{\theta_2}} \in \aleph^+$and a fuzzy Lipschitz operator $S \in \mathscr{FW}^L(X,G)$, hence $T \in \mathscr{FW}_{\theta_2}^L(X,F)$

To verify $(\mathbf{b})$, let $T \in \left(\mathscr{FW}_{\theta_1}^L\right)^{\theta_2}(X,F)$. Then

$$\begin{aligned} &\|Tx - Ty\| \preceq \xi^{\theta_2} \odot \|Sx - Sy\|^{1-\theta_2} \\ &\odot \overline{\mathbb{D}}_X(x,y)^{\theta_2}, \forall x,y \in X \end{aligned} \tag{5}$$

holds for a fuzzy real number $\tilde{\xi} \in \aleph^+$, suitable complete fuzzy normed space G and a fuzzy Lipschitz operator $S \in \mathscr{FW}_{\theta_1}^L(X,G)$ and

$$\begin{aligned} &\|Sx - Sy\| \preceq \gamma^{\theta_1} \odot \|Rx - Ry\|^{1-\theta_1} \\ &\odot \overline{\mathbb{D}}_X(x,y)^{\theta_1}, \forall x,y \in X \end{aligned} \tag{6}$$

holds for a fuzzy real number $\gamma \in \aleph^+$, suitable complete fuzzy normed space G and a fuzzy Lipschitz operator $R \in \mathscr{FW}^L(X,G)$. From Eqs. (5) and (6) we have

$$\begin{aligned} \|Tx - Ty\| &\preceq \xi^{\theta_2} \odot \gamma^{\theta_1\cdot(1-\theta_2)} \odot \|Rx - Ry\|^{(1-\theta_1)\cdot(1-\theta_2)} \\ &\odot \overline{\mathbb{D}}_X(x,y)^{\theta_1\cdot(1-\theta_2)} \odot \overline{\mathbb{D}}_X(x,y)^{\theta_2} \\ &\preceq \xi^{\theta_2+\theta_1-\theta_1\cdot\theta_2} \odot \|Rx - Ry\|^{1-\theta_2-\theta_1+\theta_1\cdot\theta_2} \\ &\odot \overline{\mathbb{D}}_X(x,y)^{\theta_2+\theta_1-\theta_1\cdot\theta_2}, \end{aligned}$$

hence $T \in \mathscr{FW}_{\theta_1+\theta_2-\theta_1\cdot\theta_2}^L(X,F)$.

Remark 3.1. Definition 3.2 can be generalized as follows. Let $0 \le \theta < 1$ and let $\mathscr{FW}_\theta^L$ and $\mathscr{FB}_\theta^L$ be fuzzy Lipschitz ideals. A fuzzy Lipschitz operator T from X into F belongs to $\left(\mathscr{F}_\theta^L, \mathscr{FB}_\theta^L\right)(X,F)$ if there exist fuzzy real numbers $\tilde{\xi}, \gamma \in \aleph^+$, complete fuzzy normed spaces G_1, G_2, and fuzzy Lipschitz operators $S_1 \in \mathscr{FW}_\theta^L(X,G_1), S_2 \in \mathscr{FB}_\theta^L(X,G_2)$ such that

$$\begin{aligned} &\|Tx - Ty\| \preceq \xi^\theta \odot \|S_1 x - S_1 y\|^{1-\theta} \odot \gamma^{1-\theta} \\ &\odot \|S_2 x - S_2 y\|^\theta, \forall x,y \in X. \end{aligned}$$

3.2. Absolutely fuzzy Lipschitz (p,θ)-summing maps

Definition 3.3. Let $1 < p < \infty$. A fuzzy Lipschitz map S from $(X, \overline{\mathbb{D}}_X, x_0)$ into $(Y, \overline{\mathbb{D}}_Y, y_0)$ is called absolutely fuzzy Lipschitz (p,θ)-summing if there is a fuzzy real number $\xi \in \aleph^+$such that for all $\left(x_j\right)_{j=1}^m, \left(y_j\right)_{j=1}^m$ in X and $m \in \mathcal{N}$, the partial ordering

$$\begin{aligned} &\left[\sum_{j=1}^{m} \bar{D}_Y\left(Sx_j, Sy_j\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}} \preceq \xi \odot \\ &\sup_{\mathscr{F}\,\mathrm{Lip}(f)\preceq 1} \left[\sum_{j=1}^{m} \left(\left|fx_j - fy_j\right|_\sim^{1-\theta} \odot \overline{\mathbb{D}}_X\left(x_j, y_j\right)^\theta\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}} \end{aligned} \tag{7}$$

holds. The class of all absolutely fuzzy Lipschitz (p,θ)-summing maps from $(X, \bar{D}_X, x_0)$ into $(Y, \overline{\mathbb{D}}_Y, y_0)$ is denoted by $\mathscr{FP}_{p,\theta}^L(X,Y)$. In this case, the absolutely fuzzy

Lipschitz (p,θ)-summing norm $\mathscr{F}\mathrm{P}^L_{p,\theta}(S)^{\sim}$ of S is defined by, $\left[\mathscr{F}\mathrm{P}^L_{p,\theta}(S)^{\sim}\right]_\alpha = \left[\mathscr{F}\mathrm{P}^L_{p,\theta}(S)^-_\alpha, \mathscr{F}\mathrm{P}^L_{p,\theta}(S)^+_\alpha\right]$ for all $\alpha \in (0,1]$, where and

$$\mathscr{F}\mathrm{P}^L_{p,\theta}(S)^+_\alpha := \inf\{\tilde{\zeta}^+_\alpha : (Eq.\ (7)) \text{ holds }\}.$$

Proposition 3.3. *Let $1 < p < \infty$. If $S \in \mathscr{F}\mathfrak{P}^L_{p,\theta}(X,Y)$, then $\mathscr{F}\mathrm{P}^L_{p,\theta}(S)^{\sim} \in \aleph$.*

Proof First, we prove that $\left[\mathscr{F}\mathrm{P}^L_{p,\theta}(S)^{\sim}\right]_\alpha$ is a nonempty interval for all $\alpha \in (0,1]$. Let $\alpha \in (0,1]$ and $\beta < \alpha$ and let $\left(x_j\right)_{j=1}^m, \left(y_j\right)_{j=1}^m$ in X and $m \in \mathcal{N}$. From our hypothesis we have

$$\frac{\left[\sum_{j=1}^m \mathbb{D}_Y\left(Sx_j,Sy_j\right)_\beta^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\mathscr{F}\,\mathrm{Lip}(f)\preceq\tilde{1}}\left[\sum_{j=1}^m\left(\left|fx_j-fy_j\right|_\beta^{1-\theta,-}\cdot\mathbb{D}_X\left(x_j,y_j\right)_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}} \le \zeta^-_\beta \tag{8}$$

and $\zeta^-_\beta \le \zeta^-_\alpha$. Since $\tilde{\zeta}^-_\alpha \le \zeta^+_\alpha$ we get

$$\frac{\left[\sum_{j=1}^m \mathbb{D}_Y\left(Sx_j,Sy_j\right)_\beta^{\frac{p}{1-\theta'},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\mathscr{F}\,Lip(f)\preceq 1}\left[\sum_{j=1}^m\left(\left|fx_j-fy_j\right|_\beta^{1-\theta,-}\cdot\mathbb{D}_X\left(x_j,y_j\right)_\beta^{\theta_1-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}} \le \zeta^+_\alpha$$

. Therefore

$$\begin{aligned}\mathscr{F}\mathrm{P}^L_p(S)^-_\alpha &:= \sup_{\beta<\alpha}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^m \mathbb{D}_Y(Sx_j,Sy_j)_\beta^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^m\left(\left|fx_j-fy_j\right|_\beta^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}\\ &\le \inf\{\zeta^+_\alpha : (Eq.\ (7)) \text{ holds}\}\\ &=: \mathscr{F}\mathrm{P}^L_{p,\theta}(S)^+_\alpha.\end{aligned}$$

Now we prove that $[\mathscr{F}\mathrm{P}^L_{p,\theta}(S)^{\sim}]_\alpha$ satisfies the conditions of [11]:

(i) Let $0 < \alpha_1 \le \alpha_2 \le 1$. To show that $[\mathscr{F}\mathrm{P}^L_{p,\theta}(S)^{\sim}]_{\alpha_2} \subset [\mathscr{F}\mathrm{P}^L_{p,\theta}(S)^{\sim}]_{\alpha_1}$. We have

$$\begin{aligned}\mathscr{F}\mathrm{P}^L_{p,\theta}(S)^-_{\alpha_1} &:= \sup_{\beta<\alpha_1}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^m \mathbb{D}_Y(Sx_j,Sy_j)_\beta^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^m\left(\left|fx_j-fy_j\right|_\beta^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}\\ &\le \sup_{\beta<\alpha_2}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^m \mathbb{D}_Y(Sx_j,Sy_j)_\beta^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^m\left(\left|fx_j-fy_j\right|_\beta^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}} := \mathscr{F}\mathrm{P}^L_{p,\theta}(S)^-_{\alpha_2}.\end{aligned}$$

Since $0 < \alpha_1 \le \alpha_2 \le 1$ we obtain $\zeta^+_{\alpha_2} \le \zeta^+_{\alpha_1}$ and then $\mathscr{F}\mathrm{P}^L_{p,\theta}(S)^+_{\alpha_2} := \inf\{\zeta^+_{\alpha_2} : (Eq.\ (7)) \text{ holds }\}$
$\le \inf\{\zeta^+_{\alpha_1} : (Eq.\ (7)) \text{ holds }\}$
$:= \mathscr{F}\mathrm{P}^L_{p,\theta}(S)^+_{\alpha_1}$

(ii) Let $(\alpha_k)_{k\in\mathcal{N}}$ be an increasing sequence in $(0,1]$ converging to α. To show that
$[\lim\limits_{k\to\infty}\mathscr{F}\mathrm{P}^L_{p,\theta}(S)^-_{\alpha_k}, \lim\limits_{k\to\infty}\mathscr{F}\mathrm{P}^L_{p,\theta}(S)^+_{\alpha_k}] = [\mathscr{F}\mathrm{P}^L_{p,\theta}(S)^-_\alpha, \mathscr{F}\mathrm{P}^L_{p,\theta}(S)^+_\alpha]$. We have $\alpha_k \le \alpha_{k+1} \le \alpha$ and thus

$$\begin{aligned}&\sup_k\sup_{\beta<\alpha_k}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^m \mathbb{D}_Y(Sx_j,Sy_j)_\beta^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^m\left(\left|fx_j-fy_j\right|_\beta^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}\\ &\le \sup_{\beta<\alpha}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^m \mathbb{D}_Y(Sx_j,Sy_j)_\beta^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^m\left(\left|fx_j-fy_j\right|_\beta^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}.\end{aligned} \tag{9}$$

Let $\epsilon > 0$. Then there exist $\beta_0 < \alpha$ such that

$$\begin{aligned}&\sup_{\beta<\alpha}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^m \mathbb{D}_Y(Sx_j,Sy_j)_\beta^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^m\left(\left|fx_j-fy_j\right|_\beta^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}} - \epsilon\\ &< \sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^m \mathbb{D}_Y(Sx_j,Sy_j)_{\beta_0}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^m\left(\left|fx_j-fy_j\right|_{\beta_0}^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_{\beta_0}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}.\end{aligned}$$

Since $\alpha_k \nearrow \alpha$, there is $0 < k_0$ such that $\beta_0 < \alpha_{k_0} \le \alpha$. Then

$$\begin{aligned}&\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^m \mathbb{D}_Y(Sx_j,Sy_j)_{\beta_0}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^m\left(\left|fx_j-fy_j\right|_{\beta_0}^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_{\beta_0}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}\\ &\le \sup_{\beta<\alpha_{k_0}}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^m \mathbb{D}_Y(Sx_j,Sy_j)_\beta^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^m\left(\left|fx_j-fy_j\right|_\beta^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}\\ &\le \sup_k\sup_{\beta<\alpha_k}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^m \mathbb{D}_Y(Sx_j,Sy_j)_\beta^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^m\left(\left|fx_j-fy_j\right|_\beta^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}.\end{aligned}$$

Therefore

$$\begin{aligned}&\sup_{\beta<\alpha}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^m \mathbb{D}_Y(Sx_j,Sy_j)_\beta^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^m\left(\left|fx_j-fy_j\right|_\beta^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}} - \epsilon\\ &< \sup_k\sup_{\beta<\alpha_k}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^m \mathbb{D}_Y(Sx_j,Sy_j)_\beta^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^m\left(\left|fx_j-fy_j\right|_\beta^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}.\end{aligned}$$

As $\epsilon \longrightarrow 0$, we have

$$\sup_{\beta<\alpha}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^{m}\mathbb{D}_Y(Sx_j,Sy_j)_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^{m}\left(\left|fx_j-fy_j\right|_{\beta}^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}$$
$$\leq\sup_{k}\sup_{\beta<\alpha_k}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^{m}\mathbb{D}_Y(Sx_j,Sy_j)_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^{m}\left(\left|fx_j-fy_j\right|_{\beta}^{1-\theta,-}\cdot\mathbb{D}_Y(x_j,y_j)_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}. \tag{10}$$

From Inequalities (9) and (10). We obtain

$$\lim_{k\to\infty}\sup_{\beta<\alpha_k}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^{m}\mathbb{D}_Y(Sx_j,Sy_j)_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^{m}\left(\left|fx_j-fy_j\right|_{\beta}^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}$$
$$=\sup_{k}\sup_{\beta<\alpha_k}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^{m}\mathbb{D}_Y(Sx_j,Sy_j)_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^{m}\left(\left|fx_j-fy_j\right|_{\beta}^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}$$
$$=\sup_{\beta<\alpha}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^{m}\mathbb{D}_Y(Sx_j,Sy_j)_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^{m}\left(\left|fx_j-fy_j\right|_{\beta}^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}.$$

Since $\alpha_k\leq\alpha$, we have

$$\begin{aligned}\mathscr{F}\mathsf{P}_{p,\theta}^{L}(S)_{\alpha}^{+}&:=\inf\left\{\xi_{\alpha}^{+}:(Eq.\ (7))\text{ holds}\right\}\\&\leq\inf\left\{\xi_{\alpha_k}^{+}:(Eq.\ (7))\text{ holds}\right\}\\&=:\mathscr{F}\mathsf{P}_{p,\theta}^{L}(S)_{\alpha_k}^{+},\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{k\to\infty}\inf\left\{\xi_{\alpha_k}^{+}:(Eq.\ (7))\text{ holds}\right\}&=\inf_{k}\inf\left\{\xi_{\alpha_k}^{+}:(Eq.\ (7))\text{ holds}\right\}\\&=\inf\left\{\xi_{\alpha}^{+}:(Eq.\ (7))\text{ holds}\right\}.\end{aligned}$$

(iii) To prove that $-\infty<\mathscr{F}\mathsf{P}_{p,\theta}^{L}(S)_{\alpha}^{-}\leq\mathscr{F}\mathsf{P}_{p,\theta}^{L}(S)_{\alpha}^{+}<\infty$, for all $\alpha\in(0,1]$.

Since $0\leq\frac{\left[\sum_{j=1}^{m}\mathbb{D}_Y(Sx_j,Sy_j)_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum_{j=1}^{m}\left(\left|fx_j-fy_j\right|_{\beta}^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}$ for all $x_j\neq y_j\in X$ and all $\beta\in(0,1]$. Then $0\leq\mathscr{F}\mathsf{P}_{p,\theta}^{L}(S)_{\alpha}^{-}$. Let $\xi\in\mathscr{F}$ such that for all $(x_j)_{j=1}^{m}$, $(y_j)_{j=1}^{m}$ in X and $m\in\mathbb{N}$, Partial ordering (Eq. (7)) holds. It follows that $\xi_{\alpha}^{+}<\infty$, for all $\alpha\in(0,1]$. Hence $\mathscr{F}\mathsf{P}_{p,\theta}^{L}(S)_{\alpha}^{+}<\infty$. Thus we obtain $\mathscr{F}\mathsf{P}_{p,\theta}^{L}(S)^{\sim}$ is a fuzzy real number.

Proposition3.4. *Let $1<p<\infty$. If $S\in\mathscr{F}\mathfrak{P}_{p,\theta}^{L}(X,Y)$, then*

$$\left[\sum_{j=1}^{m}\overline{\mathbb{D}}_Y(Sx_j,Sy_j)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}\preceq\xi$$
$$\odot\sup_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum_{j=1}^{m}\left(\left|fx_j-fy_j\right|_{\sim}^{1-\theta}\odot\overline{\mathbb{D}}_X(x_j,y_j)^{\theta}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}$$

for all $(x_j)_{j=1}^{m}$, $(y_j)_{j=1}^{m}$ in X and $m\in\mathbb{N}$.

Proof Suppose that $(\beta_k)_{k\in\mathbb{N}}$ be an increasing sequence in $(0,1]$ converging to $\alpha\in(0,1]$. Since

$$\frac{\left[\sum\limits_{j=1}^{m}\mathbb{D}_Y(Sx_j,Sy_j)_{\beta_k}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^{m}\left(\left|fx_j-fy_j\right|_{\beta_k}^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_{\beta_k}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}$$
$$\leq\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum\limits_{j=1}^{m}\mathbb{D}_Y(Sx_j,Sy_j)_{\beta_k}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum\limits_{j=1}^{m}\left(\left|fx_j-fy_j\right|_{\beta_k}^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_{\beta_k}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}\leq\mathscr{F}\mathsf{P}_{p,\theta}^{L}(S)_{\alpha}^{-}.$$

Then $\left[\sum_{j=1}^{m}\mathbb{D}_Y(Sx_j,Sy_j)_{\beta_k}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}\leq\mathscr{F}\mathsf{P}_{p,\theta}^{L}(S)_{\alpha}^{-}\cdot\sup_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum_{j=1}^{m}\left(\left|fx_j-fy_j\right|_{\beta_k}^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_{\beta_k}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}.$

Since $\beta_k\nearrow\alpha$, it follows from [11] that

$$\begin{aligned}\left[\sum_{j=1}^{m}\mathbb{D}_Y(Sx_j,Sy_j)_{\alpha}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}&=\lim_{k\to\infty}\left[\sum_{j=1}^{m}\mathbb{D}_Y(Sx_j,Sy_j)_{\beta_k}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}\\&\leq\mathscr{F}\mathsf{P}_{p,\theta}^{L}(S)_{\alpha}^{-}\cdot\lim_{k\to\infty}\sup_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum_{j=1}^{m}\left(\left|fx_j-fy_j\right|_{\beta_k}^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_{\beta_k}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}\\&\leq\mathscr{F}\mathsf{P}_{p,\theta}^{L}(S)_{\alpha}^{-}\cdot\sup_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum_{j=1}^{m}\left(\left|fx_j-fy_j\right|_{\alpha}^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_{\alpha}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}.\end{aligned}$$

Then

$$\left[\sum_{j=1}^{m}\mathbb{D}_Y(Sx_j,Sy_j)_{\alpha}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}\leq\mathscr{F}\mathsf{P}_{p,\theta}^{L}(S)_{\alpha}^{-}$$
$$\cdot\sup_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum_{j=1}^{m}\left(\left|fx_j-fy_j\right|_{\alpha}^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_{\alpha}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}} \tag{11}$$

From our hypothesis we have

$$\left[\sum_{j=1}^{m}\mathbb{D}_Y(Sx_j,Sy_j)_{\alpha}^{\frac{p}{1-\theta},+}\right]^{\frac{1-\theta}{p}}\leq\xi_{\alpha}^{+}\cdot$$
$$\sup_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum_{j=1}^{m}\left(\left|fx_j-fy_j\right|_{\alpha}^{1-\theta,+}\cdot\mathbb{D}_X(x_j,y_j)_{\alpha}^{\theta,+}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}} \tag{12}$$

Then

$$\left[\sum_{j=1}^{m} \mathbb{D}_Y(Sx_j, Sy_j)_\alpha^{\frac{p}{1-\theta},+}\right]^{\frac{1-\theta}{p}} \leq \inf\{\xi_\alpha^+ : (Eq.\ (7))\ \text{holds}\}$$

$$\cdot \sup_{\mathscr{F}Lip(f)\preceq\tilde{1}} \left[\sum_{j=1}^{m}\left(\left|fx_j - fy_j\right|_\alpha^{1-\theta,+} \cdot \mathbb{D}_X(x_j,y_j)_\alpha^{\theta,+}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}.$$

From Inequalities (12) and (13), we fulfill the requirement.

Remark 3.2. *From Definition 3.3 and Proposition 3.4 we conclude that* $\mathscr{F}\mathfrak{P}_{p,\theta}^L(X,Y) \subseteq \mathscr{F}Lip(X,Y)$ *with* $\mathscr{F}Lip(S)^\sim \preceq \mathscr{F}\mathsf{P}_{p,\theta}^L(S)^\sim$.

Proposition 3.5. *Let* $1 < p < \infty$. *If* $S \in \mathscr{F}\mathfrak{P}_{p,\theta}^L(X,Y)$, *then* $\mathscr{F}\mathsf{P}_{p,\theta}^L(S)^\sim \preceq \xi$, *where* ξ *defined in* (*Eq.* (7)).

Proof Let $\alpha \in (0,1]$ and $\beta < \alpha$. From Inequality (**??**) we have

$$\mathscr{F}\mathsf{P}_{p,\theta}^L(S)_\alpha^- := \sup_{\beta<\alpha}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}} \frac{\left[\sum_{j=1}^{m}\mathbb{D}_Y(Sx_j,Sy_j)_\beta^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\mathscr{F}Lip(f)\preceq\tilde{1}}\left[\sum_{j=1}^{m}\left(\left|fx_j-fy_j\right|_\beta^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}} \leq \xi_\alpha^-$$

Since $\mathscr{F}\mathsf{P}_{p,\theta}^L(S)_\alpha^+ := \inf\{\xi_\alpha^+ : (Eq.\ (7))\ \text{holds}\} \leq \xi_\alpha^+$, then $\mathscr{F}\mathsf{P}_{p,\theta}^L(S)_\alpha^+ \leq \xi_\alpha^+$. Thus, we conclude that $\mathscr{F}\mathsf{P}_{p,\theta}^L(S)^\sim \preceq \xi$.

Proposition 3.6. *Let* $0 \leq \theta < 1$ *and* $1 < p < \infty$. *If* $T \in \mathscr{F}\mathcal{W}_\theta^L(X,F)$, *then there are a fuzzy real number* $\xi \in \aleph^+$ *and a regular probability measure* $\varkappa$ *defined on* $\mathfrak{B}_{\mathscr{F}_{X^\#}}$ *such that*

$$\|Tx-Ty\|_\sim \preceq \xi \odot \left(\int_{\mathfrak{B}_{\mathscr{F}_{X^\#}}}\left(|fx-fy|_\sim^{1-\theta}\odot\overline{\mathbb{D}}_X(x,y)^\theta\right)^{\frac{p}{1-\theta}} d\varkappa(f)\right)^{\frac{1-\theta}{p}},$$

for all x and y in X.

Proof From our hypothesis there are fuzzy real number $\xi \in \aleph^+$, complete fuzzy normed space G and fuzzy Lipschitz operator $S \in \mathscr{F}\mathcal{W}^L(X,G)$ such that

$$\|Tx-Ty\|_\sim \preceq \xi^\theta \odot \|Sx-Sy\|_\sim^{1-\theta}\odot\overline{\mathbb{D}}_X(x,y)^\theta, \forall x,y\in X. \tag{13}$$

From [10] there is a regular probability measure $\varkappa$ defined on $\mathfrak{B}_{\mathscr{F}X^\#}$ such that

$$\|Sx-Sy\|_\sim \preceq \mathscr{F}\mathsf{P}_p(S)\odot\left(\int_{\mathfrak{B}_{\mathscr{F}_{X^\#}}}|fx-fy|_\sim^p\, d\varkappa(f)\right)^{\frac{1}{p}}, \forall x,y\in X.$$

$$\|Tx-Ty\|_\sim \preceq \xi^\theta\odot\mathscr{F}\mathsf{P}_p(S)\odot\left(\int_{\mathfrak{B}_{\mathscr{F}_{X^\#}}}|fx-fy|_\sim^p\, d\varkappa(f)\right)^{\frac{1}{p}}\odot\overline{\mathbb{D}}_Y(x,y)^\theta, \forall x,y\in X.$$

We now prove the fundamental characterization of absolutely fuzzy Lipschitz (p,θ)-summing maps.

Theorem 3.1. *Let* $1 < p < \infty$. *A fuzzy Lipschitz map* $S \in \mathscr{F}\mathfrak{P}_{p,\theta}^L(X,Y)$ *if and only if there are a fuzzy real number* $\xi \in \aleph^+$ *and a regular probability measure* $\varkappa$ *defined on* $\mathfrak{B}_{\mathscr{F}_{X^\#}}$ *such that*

$$\overline{\mathbb{D}}_Y(Sx,Sy) \preceq \xi\odot\left(\int_{\mathfrak{B}_{\mathscr{F}_{X^\#}}}\left(|fx-fy|_\sim^{1-\theta}\odot\overline{\mathbb{D}}_X(x,y)^\theta\right)^{\frac{p}{1-\theta}} d\varkappa(f)\right)^{\frac{1-\theta}{p}},$$

for all x and y in X.

Proof To prove the necessary condition. Suppose that $m\in\mathcal{N}$, $x_1,\ldots,x_m$, $y_1,\ldots,y_m$ in X and $\alpha\in(0,1]$,

$$\sum_{j=1}^{m}\mathbb{D}_Y(Sx_j,Sy_j)_\alpha^{\frac{p}{1-\theta},-} \leq \xi_\alpha^{\frac{p}{1-\theta},-} \tag{14}$$

$$\cdot\sum_{j=1}^{m}\int_{\mathfrak{B}_{\mathscr{F}X^\#}}\left(\left|fx_j-fy_j\right|_\alpha^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_\alpha^{\theta,-}\right)^{\frac{p}{1-\theta}} d\varkappa(f)$$

$$\leq \xi_\alpha^{\frac{p}{1-\theta},-}\cdot\int_{\mathfrak{B}_{\mathscr{F}X^\#}}\sum_{j=1}^{m}\left(\left|fx_j-fy_j\right|_\alpha^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_\alpha^{\theta,-}\right)^{\frac{p}{1-\theta}} d\varkappa(f)$$

$$\leq \xi_\alpha^{\frac{p}{1-\theta},-}\cdot\sup_{\mathscr{F}Lip(f)\preceq\tilde{1}}\sum_{j=1}^{m}\left(\left|fx_j-fy_j\right|_\alpha^{1-\theta,-}\cdot\mathbb{D}_X(x_j,y_j)_\alpha^{\theta,-}\right)^{\frac{p}{1-\theta}}. \tag{15}$$

In the same fashion of Inequality (14), we obtain

$$\sum_{j=1}^{m}\mathbb{D}_Y(Sx_j,Sy_j)_\alpha^{\frac{p}{1-\theta},+} \leq \xi_\alpha^{\frac{p}{1-\theta},+}$$

$$\cdot\sup_{\mathscr{F}Lip(f)\preceq\tilde{1}}\sum_{j=1}^{m}\left(\left|fx_j-fy_j\right|_\alpha^{1-\theta,+}\cdot\mathbb{D}_X(x_j,y_j)_\alpha^{\theta,+}\right)^{\frac{p}{1-\theta}} \tag{16}$$

From Inequalities (14) and (16), we obtain $S\in\mathscr{F}\mathfrak{P}_{p,\theta}^L(X,Y)$ with $\mathscr{F}\mathsf{P}_{p,\theta}^L(S)^\sim \preceq \xi$. Conversely, suppose that $S\in\mathscr{F}\mathfrak{P}_\theta^L(X,Y)$. Let $\mathbb{M}$ be an arbitray subset of $X\times X$ the map $\Gamma_\mathbb{M}$ on $\mathfrak{B}_{\mathscr{F}X^\#}$ by

$$\Gamma_\mathbb{M}(f)^\sim := \sum_{(x,y)\in M}\left(\mathscr{F}\mathsf{P}_{p,\theta}^L(S)^{\frac{p}{1-\theta},\sim}\right.$$

$$\left.\odot|f(x)-f(y)|_\sim^p\odot\overline{\mathbb{D}}_X(x,y)^{\frac{p\theta}{1-\theta}}\ominus\overline{\mathbb{D}}_Y(Sx,Sy)^{\frac{p}{1-\theta}}\right).$$

The fuzzy number $\Gamma_\mathbb{M}(f)^\sim$ is defined by $[\Gamma_\mathbb{M}(f)^\sim]_\alpha = [\Gamma_\mathbb{M}(f)_\alpha^-, \Gamma_\mathbb{M}(f)_\alpha^+]$, where

$$\Gamma_\mathbb{M}(f)_\alpha^- := \sum_{(x,y)\in\mathbb{M}}\left(\mathscr{F}\mathsf{P}_{p,\theta}^L(S)_\alpha^{\frac{p}{1-\theta},-}\cdot|f(x)-f(y)|_\alpha^{p,-}\right.$$

$$\left.\cdot\mathbb{D}_X(x,y)_\alpha^{\frac{p\theta}{1-\theta},-}-\mathbb{D}_Y(Sx,Sy)_\alpha^{\frac{p}{1-\theta},-}\right) \tag{17}$$

and

$$\Gamma_\mathbb{M}(f)_\alpha^+ := \sum_{(x,y)\in\mathbb{M}}\left(\mathscr{F}\mathsf{P}_{p,\theta}^L(S)_\alpha^{\frac{p}{1-\theta},+}\cdot|f(x)-f(y)|_\alpha^{p,+}\right.$$

$$\left.\cdot\mathbb{D}_X(x,y)_\alpha^{\frac{p\theta}{1-\theta},+}-\mathbb{D}_Y(Sx,Sy)_\alpha^{\frac{p}{1-\theta},+}\right). \tag{18}$$

For arbitrary order pair $(x,y) \in \mathbb{M}$ the maps $L_{(x,y)} : \mathfrak{B}_{\mathscr{F}X^{\#}} \longrightarrow \mathcal{R}$, $L_{(x,y)}(f) := |f(x)-f(y)|^p$, are continuous on $\mathfrak{B}_{\mathscr{F}X^{\#}}$, plainly the rules $\Gamma_{\mathbb{M}}(\cdot)^-_\alpha$ and $\Gamma_{\mathbb{M}}(\cdot)^+_\alpha$ (17) and (18) in $C(\mathfrak{B}_{\mathscr{F}X^{\#}})$, respectively. Thus $S \in \mathscr{F}\mathfrak{P}^L_{p,\theta}(X,Y)$ and $\Gamma_{\mathbb{M}}(f)^-_\alpha \preceq \Gamma_{\mathbb{M}}(f)^+_\alpha$ thereby $\sup_{\|f\|^\sim \preceq \tilde{1}} \Gamma_{\mathbb{M}}(f)^-_\alpha \geq 0$ and $\sup_{\|f\|^\sim \preceq \tilde{1}} \Gamma_{\mathbb{M}}(f)^+_\alpha \geq 0$. Let $\mathcal{B}^-_\alpha := \left\{\Gamma^-_{\mathbb{M},\alpha} : \mathbb{M} \subset X \times X\right\}$ and $\mathcal{B}^+_\alpha := \left\{\Gamma^+_{\mathbb{M},\alpha} : M \subset X \times X\right\}$ be the convex subsets of $\mathfrak{B}_{\mathscr{F}X^{\#}}$ for every $\alpha \in (0,1]$. Consider the open convex subset

$$\mathcal{A} := \left\{ \Gamma \in C(\mathfrak{B}_{\mathscr{F}X^{\#}}) : \sup_{\mathscr{F}Lip(f) \preceq \tilde{1}} \Gamma(f) < 0 \right\}$$

of $C(\mathfrak{B}_{\mathscr{F}X^{\#}})$. The sets $\mathcal{A} \cap \mathcal{B}^-_\alpha$ and $\mathcal{A} \cap \mathcal{B}^+_\alpha$ are non-void for all $\alpha \in (0,1]$ one gets

$$\langle \varkappa, \Gamma \rangle < r_1 \leq \left\langle \varkappa, \Gamma^-_{\mathbb{M},\alpha} \right\rangle, \forall\, (\Gamma, \Gamma^-_{\mathbb{M},\alpha}) \in \mathcal{A} \times \mathcal{B}^-_\alpha, \quad (19)$$

and

$$\langle \varkappa, \Gamma \rangle < r_2 \leq \left\langle \varkappa, \Gamma^+_{\mathbb{M},\alpha} \right\rangle, \forall\, (\Gamma, \Gamma^+_{\mathbb{M},\alpha}) \in \mathcal{A} \times \mathcal{B}^+_\alpha. \quad (20)$$

From (19) we obtain

$$0 \leq \left\langle \varkappa, \Gamma^-_{\{x,y\},\alpha} \right\rangle = \int_{\mathfrak{B}_{\mathscr{F}X^{\#}}} \left(\mathscr{F}\mathsf{P}^L_{p,\theta}(S)^{\frac{p}{1-\theta},-}_\alpha \cdot |f(x)-f(y)|^{p,-}_\alpha \cdot \mathbb{D}_X(x,y)^{\frac{p\theta}{1-\theta},-}_\alpha - \mathbb{D}_Y(Sx,Sy)^{\frac{p}{1-\theta},+}_\alpha \right) d\varkappa(f),$$

$\forall\ x,y \in X$. Since $\mathbb{D}_Y(Sx,Sy)^{\frac{p}{1-\theta},-}_\alpha \leq \mathbb{D}_Y(Sx,Sy)^{\frac{p}{1-\theta},+}_\alpha, \forall\ \alpha \in (0,1]$ we obtain

$$\mathbb{D}_Y(Sx,Sy)^{\frac{p}{1-\theta},-}_\alpha \leq \mathscr{F}\mathsf{P}^L_{p,\theta}(S)^{\frac{p}{1-\theta},-}_\alpha \cdot \int_{\mathfrak{B}_{\mathscr{F}X^{\#}}} |f(x)-f(y)|^{p,-}_\alpha \cdot \mathbb{D}_X(x,y)^{\frac{p\theta}{1-\theta},+}_\alpha d\varkappa(f), \quad (21)$$

$\forall\ x,y \in X$. Also from (20) we have

$$0 \leq \left\langle \varkappa, \Gamma^+_{\{x,y\},\alpha} \right\rangle = \int_{\mathfrak{B}_{\mathscr{F}X^{\#}}} \left(\mathscr{F}\mathsf{P}^L_{p,\theta}(S)^{\frac{p}{1-\theta},+}_\alpha \cdot |f(x)-f(y)|^{p,+}_\alpha \cdot \mathbb{D}_X(x,y)^{\frac{p\theta}{1-\theta},+}_\alpha - \mathbb{D}_Y(Sx,Sy)^{\frac{p}{1-\theta},-}_\alpha \right) d\varkappa(f),$$

$\forall\ x,y \in X$. Since $\mathscr{F}\mathsf{P}^L_p(S)^{p,-}_\alpha \leq \mathscr{F}\mathsf{P}^L_p(S)^{p,+}_\alpha, \forall\ \alpha \in (0,1]$ we obtain

$$\mathbb{D}_Y(Sx,Sy)^{\frac{p}{1-\theta},+}_\alpha \leq \mathscr{F}\mathsf{P}^L_{p,\theta}(S)^{\frac{p}{1-\theta},+}_\alpha \cdot \int_{\mathfrak{B}_{\mathscr{F}X^{\#}}} |f(x)-f(y)|^{p,+}_\alpha \cdot \mathbb{D}_X(x,y)^{\frac{p\theta}{1-\theta},+}_\alpha d\varkappa(f), \forall\ x,y \in X. \quad (22)$$

From Inequalities (21) and (22), we get

$$\overline{\mathbb{D}}_Y(Sx,Sy) \preceq \xi \odot \left(\int_{\mathfrak{B}_{\mathscr{F}X^{\#}}} \left(|fx-fy|^{1-\theta}_\sim \odot \overline{\mathbb{D}}_X(x,y)^\theta \right)^{\frac{p}{1-\theta}} d\varkappa(f) \right)^{\frac{1-\theta}{p}},$$

$\forall\ x,y \in X$.

From the result mentioned above the following inclusion is obvious.

Proposition 3.7. *If* $p_1 \leq p_2$, *then* $\left[\mathscr{F}\mathfrak{P}^L_{p_1,\theta}, \mathscr{F}\mathsf{P}^L_{p_1,\theta}(\cdot)^\sim\right] \subseteq \left[\mathscr{F}\mathfrak{P}^L_{p_2,\theta}, \mathscr{F}\mathsf{P}^L_{p_2,\theta}(\cdot)^\sim\right]$.

4. Open problems

1. Prove or disprove $\mathscr{F}\mathsf{P}^L_{p,\theta}(T_1 \circ T_2)^\sim \preceq \mathscr{F}\mathsf{P}^L_{r,\theta}(T_1)^\sim \odot \mathscr{F}\mathsf{P}^L_{s,\theta}(T_2)^\sim$ true for arbitrary absolutely fuzzy Lipschitz (r,θ)-summing operators T_1, absolutely fuzzy Lipschitz (s,θ)-summing operators T_2 and $\frac{1}{p} \leq (\frac{1}{r} + \frac{1}{s}) \wedge 1$?

2. Describe the dual space of $\mathscr{F}\mathfrak{P}^L_{p,\theta}(X,F)$ whenever F is a complete fuzzy normed space and X is finite fuzzy pointed metric space ?

3. What is the algorithm for calculating absolute fuzzy Lipschitz (p,θ)-summing norm of fuzzy Lipschitz maps between finite fuzzy pointed metric spaces exactly ?

5. Disclosure statement

No potential conflict of interest was reported by the authors.

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