On Fuzzy Fréchet Contractions Governed By Rational Expressions And Related Applications

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Abstract. This work explores a specific category of contraction mappings characterized by rational- type within the framework of fuzzy Fréchet spaces(FFS). By utilizing the triangular inequality associated with fuzzy semi-norms, we establish fixed point results relevant to these mappings. As an application, we analyze an integral operator formulated in the FFS setting and prove the existence and uniqueness of its solution. This study concludes with an integral-type example that illustrates the effectiveness of rational fuzzy Fréchet contractions in solving operator equations.

Keywords: Rational type fuzzy Fréchet-contraction; Fuzzy Fréchet space; Fixed point

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1. Introduction

The Banach contraction principle ($B-cp$) asserts that a self-mapping on a complete metric space, provided it satisfies a contraction condition, possesses a unique fixed point [1]. Several writers have applied various generalizations of the ($B-cp$) theorem because of its applicability. However, one of the most significant fields of modern mathematics is functional analysis. It is crucial. In representation theory, probability, differential equation theory, and the study of numerous attributes of various spaces, including metric space, Hilbert space, Banach space, and others, see ([2–8]). With the publication of Zadeh's seminal paper [9], which introduced the brilliant idea of a fuzzy set, a great number of mathematicians become cognizant of the numerous ways in which the new fuzzy framework could be used to extend the classical conclusions and their infinite possibilities. The notation of fuzzy semi-norms was introduced by Katsaras [10] in 1984. Sadepqi and Solaty [11] expanded on this idea later, in 2007. Two such methods were used in 2021 by Ahmed Ghanawi and Al. Nafie [12, 13] developed the idea of fuzzy Fréchet space.

As mentioned above, this motivates us in this study to address rational-type contraction mappings within fuzzy Fréchet spaces and establish several specialized fixed point results under corresponding contraction conditions, accompanied by illustrative examples. The goal of our work is to expand the applicability of fuzzy contraction theory and contribute significantly to the understanding of fixed point phenomena in such spaces. To support the theoretical findings, an application of integral type aligned with the approach of Jabeen et al. [14] is presented. This application highlights the relevance of the results by encompassing a range of nonlinear integral equations, thereby ensuring the existence and uniqueness of solutions under the proposed conditions.

Now, we present some fundamental definitions and preliminary concepts that will be essential in developing our main results.

Definition 1.1. [15] We call a mapping $* : [0,1] \times [0,1] \to [0,1]$ a continuous t-norm if the following properties hold: 1. $*$ is continuous, associative, and commutative, 2. For every $k_1 \leq k_3$ and $k_2 \leq k_4$, where k_1, k_2, k_3, and $k_4 \in [0,1], 1 * k_1 = k_1$ and $k_1 * k_2 \leq k_3 * k_4$.

The following definitions apply to the fundamental t-norm; $k_1 * k_2 = \min\{k_1 * k_2\}, k_1 * k_2 = k_1 k_2$.

Definition 1.2. [11] Let V represent a vector space across a field K. If the following requirements are met, a fuzzy set ψ in $V \times R$ is said to be a fuzzy semi-norm on $V : \forall w, z \in V$ and $\forall k_1, k_2 \in R$, if 1. $\psi(w, k_1) = 0, \forall k_1 \leq 0$ 2. $\psi(\lambda w, k_1) = \psi\left(w, \frac{k_1}{\lambda}\right), \forall \lambda \in K/\{0\}, \forall k_1 > 0$ 3. $\psi(w + z, k_1 + k_2) \geq \psi(w, k_1) * \psi(z, k_2)$ 4. $\psi(w, k_1)$ is non-decreasing w.r.t $k_1, \forall w \in V, \lim_{k_1 \to 0} \psi(w, k_1) = 0$, and, $\lim_{k_1 \to \infty} \psi(w, k_1) = 1$.

Definition 1.3. [11] If for every w $\neq 0$ in a vector space V, there is at least one $\psi \in E$ and $k_1 > 0$ such that $\psi(w, k_1) \neq 1$ then the family E of fuzzy semi-norms is said to be separating.

Definition 1.4. [12] A vector space V is referred to as fuzzy Fréchet space (FFS) if it is a fuzzy topological vector space that is complete, and its fuzzy topology τ_E is determined by a countable, separating family of fuzzy semi-norms denoted by $E = \{\psi_j\}_{j \in I}$.

The construction of fuzzy Fréchet space as well as the concepts of fuzzy convergence, fuzzy Cauchy sequence, and fuzzy continuity in FFS, are examined in [12].

2. Main results

In this section, we introduce our basic definitions in this study: the triangle property in fuzzy Fréchet space, the fuzzy Fréchet contraction, and a rational-type fuzzy Fréchet contraction. We also prove fixed-point theorems with illustrative examples.

Let V be FFS, with countable separating family of fuzzy semi-norms $E = \{\psi_j\}_{j \in I}$.

Definition 2.1. A collection of fuzzy semi-norms $E = \{\psi_j\}_{j \in I}$ is said to satisfy the triangular property whenever

$$\left(\frac{1}{\psi_j(w - y, k)} - 1\right) + \left(\frac{1}{\psi_j(y - z, k)} - 1\right) \geq \left(\frac{1}{\psi_j(w - z, k)} - 1\right) \tag{1}$$

$\forall w, z, y \in V, k > 0$, and $\forall \psi_j \in E$

Definition 2.2. A mapping $h : V \to V$ is called fuzzy Fréchet-contraction, if there exists $r \in (0, 1)$, so that

$$r\left(\frac{1}{\psi_j(w - z, k)} - 1\right) \geq \left(\frac{1}{\psi_j(hw - hz, k)} - 1\right) \tag{2}$$

For all $w, y, z \in V, k > 0$, and $\forall \psi_j \in E$.

Definition 2.3. Let $h : V \to V$ be a mapping. If there are $r, s \in [0, 1)$, such that

$$s\left(\frac{\psi_j(w - z, k)}{\psi_j(w - hw, k) * \psi_j(z - hw, 2k)} - 1\right) + r\left(\frac{1}{\psi_j(w - z, k)} - 1\right) \geq \left(\frac{1}{\psi_j(hw - hz, k)} - 1\right) \tag{3}$$

$\forall w, z, y \in V, k > 0$, and $\forall \psi_j \in E$. Under this condition, h is identified as a rational type fuzzy Fréchet-contraction. The following theorem demonstrates that if the mapping is a rational type fuzzy Fréchet-contraction, it will have a unique fixed point provided that the family $E = \{\psi_j\}_{j \in I}$ is triangular.

Theorem 2.4. Let $h : V \to V$ a rational type fuzzy Fréchet-contraction mapping with $r + s < 1$, and $E = \{\psi_j\}_{j \in I}$ is triangular. Then, in V, h has a unique fixed point.

Proof. Let $w_i \in V$ and $w_{i+1} = hw_i, i \geq 0$. Then, for $k > 0, i \geq 0$, and $\forall \psi_j \in E$,

$$\begin{aligned}
\left(\frac{1}{\psi_j(w_i - w_{i+1}, k)} - 1\right) &= \left(\frac{1}{\psi_j(hw_{i-1} - hw_i, k)} - 1\right) \\
&\leq r\left(\frac{1}{\psi_j(w_{i-1} - w_i, k)} - 1\right) \\
&+ s\left(\frac{\psi_j(w_{i-1} - w_i, k)}{\psi_j(w_{i-1} - hw_{i-1}, k) * \psi_j(w_i - hw_{i-1}, 2k)} - 1\right) \\
&= r\left(\frac{1}{\psi_j(w_{i-1} - w_i, k)} - 1\right) \\
&+ s\left(\frac{\psi_j(w_{i-1} - w_i, k)}{\psi_j(w_{i-1} - w_i, k) * \psi_j(w_i - w_i, 2k)} - 1\right) \\
&= r\left(\frac{1}{\psi_j(w_{i-1} - w_i, k)} - 1\right)
\end{aligned}$$

Then,

$$\left(\frac{1}{\psi_j(w_i - w_{i+1}, k)} - 1\right) \leq r\left(\frac{1}{\psi_j(w_{i-1} - w_i, k)} - 1\right) \tag{4}$$

Alternatively,

$$\left(\frac{1}{\psi_j(w_{i-1} - w_i, k)} - 1\right) \leq r\left(\frac{1}{\psi_j(w_{i-2} - w_{i-1}, k)} - 1\right) \tag{5}$$

By induction, for $k > 0, i \in N$, and $\forall \psi_j \in E$, we obtain that

$$\begin{aligned}
\frac{1}{\psi_j(w_i - w_{i+1}, k)} - 1\Bigg) &\leq r\left(\frac{1}{\psi_j(w_{i-1} - w_i, k)} - 1\right) \\
&\leq r^2\left(\frac{1}{\psi_j(w_{i-2} - w_{i-1}, k)} - 1\right) \\
&\leq \ldots \leq r^i\left(\frac{1}{\psi_j(w_0 - w_1, k)} - 1\right) \to 0, \text{ as } i \to \infty.
\end{aligned}$$

Therefore,

$$\lim_{i\to\infty} \psi_j\left(w_i - w_{i-1}, k\right) = 1 \tag{6}$$

$\forall k > 0$, and $\forall \psi_j \in E$. Now, we will demonstrate that $\{w_i\}_{i\in N}$ is a fuzzy Cauchy

sequence in V. Let $p \in N, \forall \psi_j \in E$, and $k > 0$, we obtain that

$$\begin{aligned}
&\psi_j(w_i - w_{i+p}, k)\\
&\geq \psi_j\left(w_i - w_{i+1}, \frac{k}{p}\right) * \psi_j\left(w_{i+1} - w_{i+2}, \frac{k}{p}\right) * \ldots\\
&* \psi_j\left(w_{i+p-1} - w_{i+p}, \frac{k}{p}\right)\\
&\to 1 * 1 * \ldots * 1 = 1
\end{aligned}$$

as $i \to \infty$. Thus $\{w_i\}_{i\in N}$ is fuzzy Cauchy sequence in V. Since V is complete, $\exists x_1 \in V$ such that $w_i \to x_1$, as $i \to \infty$, i.e.

$$\lim_{i\to\infty} \psi_j\left(w_i - x_1, k\right) = 1, \tag{7}$$

$\forall k > 0$, and $\forall \psi_j \in E$. From Eqs. (2), (6) and (7), and since $E = \left\{\psi_j\right\}_{j\in I}$ is triangular. Then, $\forall k > 0$, and $\forall \psi_j \in E$, we have

$$\begin{aligned}
&\left(\frac{1}{\psi_j\left(x_1 - hx_1, k\right)} - 1\right) \leq \left(\frac{1}{\psi_j\left(x_1 - w_{i+1}, k\right)} - 1\right)\\
&+ \left(\frac{1}{\psi_j\left(hw_i - hx_1, k\right)} - 1\right)\\
&\leq \left(\frac{1}{\psi_j\left(x_1 - w_{i+1}, k\right)} - 1\right) + r\left(\frac{1}{\psi_j\left(w_i - x_1, k\right)} - 1\right)\\
&+ s\left(\frac{\psi_j\left(w_i - x_1, k\right)}{\psi_j\left(w_i - hw_i, k\right) * \psi_j\left(x_1 - hw_i, 2k\right)} - 1\right)\\
&= \left(\frac{1}{\psi_j\left(x_1 - w_{i+1}, k\right)} - 1\right) + r\left(\frac{1}{\psi_j\left(w_i - x_1, k\right)} - 1\right)\\
&\quad + s\left(\frac{\psi_j\left(w_i - x_1, k\right)}{\psi_j\left(w_i - w_{i+1}, k\right) * \psi_j\left(x_1 - w_{i+1}, 2k\right)} - 1\right) \to 0
\end{aligned}$$

as $i \to \infty$. Then, $\psi_j\left(x_1 - hx_1, k\right) = 1, \forall k > 0$, and $\forall \psi_j \in E$. Hence, $hx_1 = x_1$ and x_1 is a fixed point of h in V. Now to show uniqueness, let $y_1 \in V$ such that $hy_1 = y_1$, then, $\forall k > 0$, and $\forall \psi_j \in E$,

$$\begin{aligned}
&\left(\frac{1}{\psi_j\left(x_1 - y_1, k\right)} - 1\right) = \left(\frac{1}{\psi_j\left(hx_1 - hy_1, k\right)} - 1\right)\\
&\leq s\left(\frac{\psi_j\left(x_1 - y_1, k\right)}{\psi_j\left(x_1 - hx_1, k\right) * \psi_j\left(y_1 - hx_1, 2k\right)} - 1\right)\\
&\quad + r\left(\frac{1}{\psi_j\left(x_1 - y_1, k\right)} - 1\right)\\
&= s\left(\frac{\psi_j\left(x_1 - y_1, k\right)}{\psi_j\left(x_1 - x_1, k\right) * \psi_j\left(y_1 - x_1, 2k\right)} - 1\right)\\
&\quad + r\left(\frac{1}{\psi_j\left(x_1 - y_1, k\right)} - 1\right)\\
&= r\left(\frac{1}{\psi_j\left(x_1 - y_1, k\right)} - 1\right)\\
&= r\left(\frac{1}{\psi_j\left(hx_1 - hy_1, k\right)} - 1\right) \leq r^2\left(\frac{1}{\psi_j\left(x_1 - y_1, k\right)} - 1\right)\\
&\leq \ldots \leq r^i\left(\frac{1}{\psi_j\left(x_1 - y_1, k\right)} - 1\right) \to 0
\end{aligned}$$

as $i \to \infty$. Then, $\psi_j\left(x_1 - y_1, k\right) = 1 \forall k > 0$, and $\forall \psi_j \in E$. Therefore, $x_1 = y_1$ and x_1 is a unique fixed point for h in V.

Example 2.5. Consider the FFS V $= [0, \infty)$ with $\psi : V \times (0, \infty) \to [0, 1]$ is triangular fuzzy semi-norm defined as follows:

$$\psi(w - z, k) = \frac{k}{k + \left|\frac{(4w - 4z)}{5}\right|}$$

$\forall w, z \in V$, and $k > 0$, with $a * b = ab, \forall a, b \in [0, 1]$. Now, we define a mapping $h : V \to V$ such that $h(w) = \begin{cases} \frac{3w}{4} & \text{if } w \in [0, 1] \\ \frac{2w}{3} + 8 & \text{if } w \in (1, \infty) \end{cases}$

Then, we obtain

$$\left(\frac{1}{\psi(hw - hz, k)} - 1\right) = \frac{3}{4}\left(\frac{1}{\psi(w - z, k)} - 1\right)$$

$\forall w, z \in V$, and $k > 0$. Thus, h is a fuzzy Fréchet-contraction. For $k > 0$, we get that

$$\begin{aligned}
&\left(\frac{\psi(w - z, k)}{\psi(w - hw, k) * \psi(z - hw, 2k)} - 1\right)\\
&\leq \left(\frac{\psi(w - z, k)}{\psi(w - hw, k) * \psi(z - w, k) * \psi(w - hw, k)} - 1\right)\\
&= \left(\frac{1}{\psi(w - hw, k) * \psi(w - hw, k)} - 1\right)\\
&= \left(\frac{1}{\psi(w - hw, k)^2} - 1\right)\\
&= \frac{2w}{5k^2}\left(\frac{w}{5} + k\right)
\end{aligned}$$

Therefore, Theorem 2.4's requirements are all met with $r = \frac{3}{4}, s = \frac{2}{9}$, and h has a fixed point $24 \in V$.

The following theorem is a generalized fuzzy Fréchet-contraction of rational type.

Theorem 2.6. Let $E = \left\{\psi_j\right\}_{j\in I}$ is triangular. If a mapping $h : V \to V$ satisfies

$$\begin{aligned}&\left(\frac{1}{\psi_j(hw-hz,k)}-1\right)\\&\leq s\left(\frac{\psi_j(w-z,k)*\psi_j(z-hz,k)}{\psi_j(w-hw,k)*\psi_j(w-hz,2k)}-1\right)\\&+r\left(\frac{1}{\psi_j(w-z,k)}-1\right)\\&+t\left(\frac{\psi_j(w-hw,k)}{\psi_j(w-hz,2k)}-1+\frac{\psi_j(z-hz,k)}{\psi_j(w-hz,2k)}-1\right)\\&+v\left(\frac{1}{\psi_j(w-hw,k)}-1+\frac{1}{\psi_j(z-hz,k)}-1\right),\end{aligned}\tag{8}$$

$\forall\psi_j \in E, \forall k > 0, \forall w, z \in V$, and $r, s, t, v \geq 0$ such that $r + s + t + v < 1$. Then, in V, h has a unique fixed point.

Proof. Let $w_0 \in V$ be fixed, and $w_{i+1} = hw_i, i \geq 0$. Then, for $k > 0, i \geq 1$, and $\psi_j \in E$,

$$\begin{aligned}&\left(\frac{1}{\psi_j\left(w_i-w_{i+1},k\right)}-1\right)=\left(\frac{1}{\psi_j\left(hw_{i-1}-hw_i,k\right)}-1\right)\\&\leq r\left(\frac{1}{\psi_j\left(w_{i-1}-w_i,k\right)}-1\right)\\&+s\left(\frac{\psi_j\left(w_{i-1}-w_i,k\right)*\psi_j\left(w_i-hw_i,k\right)}{\psi_j\left(w_{i-1}-hw_{i-1},k\right)*\psi_j\left(w_{i-1}-hw_i,2k\right)}-1\right)\\&+t\left(\frac{\psi_j\left(w_{i-1}-hw_{i-1},k\right)}{\psi_j\left(w_{i-1}-hw_i,2k\right)}-1+\frac{\psi_j\left(w_i-hw_i,k\right)}{\psi_j\left(w_{i-1}-hw_i,2k\right)}-1\right)\\&+v\left(\frac{1}{\psi_j\left(w_{i-1}-hw_{i-1},k\right)}-1+\frac{1}{\psi_j\left(w_i-hw_i,k\right)}-1\right)\\&=r\left(\frac{1}{\psi_j\left(w_{i-1}-w_i,k\right)}-1\right)\\&+s\left(\frac{\psi_j\left(w_{i-1}-w_i,k\right)*\psi_j\left(w_i-w_{i+1},k\right)}{\psi_j\left(w_{i-1}-w_i,k\right)*\psi_j\left(w_{i-1}-w_{i+1},2k\right)}-1\right)\\&+t\left(\frac{\psi_j\left(w_{i-1}-w_i,k\right)}{\psi_j\left(w_{i-1}-w_{i+1},2k\right)}-1+\frac{\psi_j\left(w_i-w_{i+1},k\right)}{\psi_j\left(w_{i-1}-w_{i+1},2k\right)}-1\right)\\&+v\left(\frac{1}{\psi_j\left(w_{i-1}-w_i,k\right)}-1+\frac{1}{\psi_j\left(w_i-w_{i+1},k\right)}-1\right)\end{aligned}$$

From (3) in definition 1.2, and after simplifying, we obtain

$$\left(\frac{1}{\psi_j\left(w_i-w_{i+1},k\right)}-1\right)\leq\lambda\left(\frac{1}{\psi_j\left(w_{i-1}-w_i,k\right)}-1\right),$$

$\forall\psi_j \in E$, and $\forall k > 0$, where $\lambda = \frac{r+s+t+v}{1-t-v} < 1$. Alternatively, $\forall\psi_j \in E$, and $\forall k > 0$, we have that

$$\left(\frac{1}{\psi_j\left(w_{i-1}-w_i,k\right)}-1\right)\leq\lambda\left(\frac{1}{\psi_j\left(w_{i-2}-w_{i-1},k\right)}-1\right)$$

By induction, $\forall\psi_j \in E$, and $\forall k > 0$, we obtain that

$$\begin{aligned}&\left(\frac{1}{\psi_j\left(w_i-w_{i+1},k\right)}-1\right)\leq\lambda\left(\frac{1}{\psi_j\left(w_{i-1}-w_i,k\right)}-1\right)\\&\leq\lambda^2\left(\frac{1}{\psi_j\left(w_{i-2}-w_{i-1},k\right)}-1\right)\leq\ldots\\&\leq\lambda^i\left(\frac{1}{\psi_j\left(w_0-w_1,k\right)}-1\right)\to 0,\end{aligned}\tag{9}$$

as $i \to \infty$. Therefore, $\lim_{i\to\infty}\psi_j\left(w_i - w_{i+1}, k\right) = 1, \forall k > 0, i \geq 1$, and $\psi_j \in E$. Now, we will demonstrate that $\{w_i\}_{i\in N}$ is a fuzzy Cauchy sequence in V. Let $p \in N$, then $\forall k > 0, i \geq 1$, and $\forall\psi_j \in E$, we obtain that

$$\begin{aligned}&\psi_j\left(w_i-w_{i+p},k\right)\geq\psi_j\left(w_i-w_{i+1},\frac{k}{p}\right)\\&*\psi_j\left(w_{i+1}-w_{i+2},\frac{k}{p}\right)*\ldots*\\&\psi_j\left(w_{i+p+1}-w_{i+p},\frac{k}{p}\right)\to 1*1*\ldots*1=1,\end{aligned}$$

as $i \to \infty$. Thus, $\{w_i\}_{i\in N}$ is a fuzzy Cauchy sequence in V. Since V is a complete, $\exists x_1 \in V$ such that $w_i \to x_1$, as $i \to \infty$, i.e.

$$\lim_{i\to\infty}\psi_j\left(w_i-x_1,k\right)=1,\tag{10}$$

$\forall k > 0$, and $\forall\psi_j \in E$. Since $E = \left\{\psi_j\right\}_{j\in I}$ is triangular, then

$$\begin{aligned}&\left(\frac{1}{\psi_j\left(x_1-hx_1,k\right)}-1\right)\leq\left(\frac{1}{\psi_j\left(x_1-w_{i+1},k\right)}-1\right)\\&+\left(\frac{1}{\psi_j\left(w_{i+1}-hx_1,k\right)}-1\right)\end{aligned}\tag{11}$$

$\forall k > 0$, and $\forall\psi_j \in E$. From Eqs. (8) to (10), $\forall k > 0$, and

$\forall \psi_j \in E$, we obtain

$$\left(\frac{1}{\psi_j(w_{i+1}-hx_1,k)}-1\right)$$
$$=\left(\frac{1}{\psi_j(hw_i-hx_1,k)}-1\right)\leq r\left(\frac{1}{\psi_j(w_i-x_1,k)}-1\right)$$
$$+s\left(\frac{\psi_j(w_i-x_1,k)*\psi_j(x_1-hx_1,k)}{\psi_j(w_i-hw_i,k)*\psi_j(w_i-hx_1,2k)}-1\right)$$
$$+t\left(\frac{\psi_j(w_i-hw_i,k)}{\psi_j(w_i-hx_1,2k)}-1+\frac{\psi_j(x_1-hx_1,k)}{\psi_j(w_i-hx_1,2k)}-1\right)$$
$$+v\left(\frac{1}{\psi_j(w_i-hw_i,k)}-1+\frac{1}{\psi_j(x_1-hx_1,k)}-1\right)$$
$$=s\left(\frac{\psi_j(w_i-x_1,k)*\psi_j(x_1-hx_1,k)}{\psi_j(w_i-w_{i+1},k)*\psi_j(w_i-hx_1,2k)}-1\right)$$
$$+r\left(\frac{1}{\psi_j(w_i-x_1,k)}-1\right)$$
$$+t\left(\frac{\psi_j(w_i-w_{i+1},k)}{\psi_j(w_i-hx_1,2k)}-1+\frac{\psi_j(x_1-hx_1,k)}{\psi_j(w_i-hx_1,2k)}-1\right)$$
$$+v\left(\frac{1}{\psi_j(w_i-w_{i+1},k)}-1+\frac{1}{\psi_j(x_1-hx_1,k)}-1\right)$$

$$\left(\frac{1}{\psi_j(w_{i+1}-hx_1,k)}-1\right)\leq r\left(\frac{1}{\psi_j(w_i-x_1,k)}-1\right)$$
$$+s\left(\frac{\psi_j(w_i-x_1,k)*\psi_j(x_1-hx_1,k)}{\psi_j(w_i-w_{i+1},k)*\psi_j(w_i-x_1,k)*\psi_j(x_1-hx_1,k)}-1\right)$$
$$+t\left(\frac{\psi_j(w_i-w_{i+1},k)}{\psi_j(w_i-x_1,k)*\psi_j(x_1-hx_1,k)}-1\right.$$
$$\left.+\frac{\psi_j(x_1-hx_1,k)}{\psi_j(w_i-x_1,k)*\psi_j(x_1-hx_1,k)}-1\right)$$
$$+v\left(\frac{1}{\psi_j(w_i-w_{i+1},k)}-1+\frac{1}{\psi_j(x_1-hx_1,k)}-1\right)$$
$$\to(t+v)\left(\frac{1}{\psi_j(x_1-hx_1,k)}-1\right),$$

From (3) in definition 1.2, $\forall k>0$, and $\forall \psi_j \in E$. we obtain that as $i\to\infty$. Thus, $\forall k>0$, and $\forall \psi_j \in E$. we have that

$$\limsup_{i\to\infty}\left(\frac{1}{\psi_j(w_{i+1}-hx_1,k)}-1\right)\leq(t+v)\left(\frac{1}{\psi_j(x_1-hx_1,k)}-1\right) \tag{12}$$

If $i\to\infty$, from Eqs. (10) to (12), we obtain that

$$\left(\frac{1}{\psi_j(x_1-hx_1,k)}-1\right)\leq(t+v)\left(\frac{1}{\psi_j(x_1-hx_1,k)}-1\right), \tag{13}$$

$\forall k>0, (t+v)<1$ and $\forall \psi_j \in E$. Therefore, $\psi_j(x_1-hx_1,k)=1, \forall k>0$, and $\forall \psi_j \in E$. This mean, $x_1=hx_1$ and x_1 is a fixed point of h.

Now to show uniqueness, let $u\in V$ such that $hu=u$. Then, $\forall k>0$, and $\forall \psi_j \in E$, we get

$$\left(\frac{1}{\psi_j(x_1-u,k)}-1\right)=\left(\frac{1}{\psi_j(hx_1-hu,k)}-1\right)$$
$$\leq s\left(\frac{\psi_j(x_1-u,k)*\psi_j(u-hu,k)}{\psi_j(x_1-hx_1,k)*\psi_j(x_1-hu,2k)}-1\right)$$
$$+r\left(\frac{1}{\psi_j(x_1-u,k)}-1\right)$$
$$+t\left(\frac{\psi_j(x_1-hx_1,k)}{\psi_j(x_1-hu,2k)}-1+\frac{\psi_j(u-hu,k)}{\psi_j(x_1-hu,2k)}-1\right)$$
$$+v\left(\frac{1}{\psi_j(x_1-hx_1,k)}-1\right.$$
$$\left.+\frac{1}{\psi_j(u-hu,k)}-1\right)$$
$$=s\left(\frac{\psi_j(x_1-u,k)}{\psi_j(x_1-hu,2k)}-1\right)+r\left(\frac{1}{\psi_j(x_1-u,k)}-1\right)$$
$$+t\left(\frac{1}{\psi_j(x_1-u,2k)}-1\right.$$
$$\left.+\frac{1}{\psi_j(x_1-u,2k)}-1\right)$$
$$=s\left(\frac{\psi_j(x_1-u,k)}{\psi_j(x_1-u,k)*\psi_j(u-u,k)}-1\right)$$
$$+r\left(\frac{1}{\psi_j(x_1-u,k)}-1\right)$$
$$+t\left(\frac{1}{\psi_j(x_1-u,k)*\psi_j(u-u,k)}-1\right.$$
$$\left.+\frac{1}{\psi_j(x_1-u,k)*\psi_j(u-u,k)}-1\right)$$
$$=\left(\frac{1}{\psi_j(x_1-u,k)}-1\right)(r+2t)$$
$$=\left(\frac{1}{\psi_j(hx_1-hu,k)}-1\right)(r+2t)$$
$$\leq\left(\frac{1}{\psi_j(x_1-u,k)}-1\right)(r+2t)^2\leq\ldots$$
$$\leq\left(\frac{1}{\psi_j(x_1-u,k)}-1\right)(r+2t)^i\to 0,$$

as $i\to\infty$, if $(r+2t)<1$. Then, $\psi_j(x_1-u,k)=1, \forall k>0$, and $\forall \psi_j \in E$. Hence, $x_1=u$ and x_1 is a unique fixed point of h in V.

Example 2.7. Consider the FFS $V=[0,\infty)$ with $\psi: V\times(0,\infty)\to[0,1]$ is triangular fuzzy semi-norm defined as follows:

$$\psi(w-z,k)=\frac{k}{k+\left|\frac{(w-z)}{2}\right|}$$

$\forall w,z\in V$, and $k>0$, with $a*b=ab,\forall a,b\in[0,1]$. Now, we define a mapping $h:V\to V$ such that $h(w)=\begin{cases}\frac{3w}{7} & \text{if } w\in[0,1]\\ \frac{3w}{4}+1 & \text{if } w\in(1,\infty)\end{cases}$ Then, we obtain

$$\left(\frac{1}{\psi(hw-hz,k)}-1\right)=\frac{3}{7}\left(\frac{1}{\psi(w-z,k)}-1\right)$$

$\forall w,z\in V$, and $k>0$. Thus, h is a fuzzy Fréchet-contraction. For $k>0$, we get that

$$\left(\frac{\psi(w-z,k)*\psi(z-hz,k)}{\psi(w-hw,k)*\psi(w-hz,2k)}-1\right)$$
$$\leq\left(\frac{1}{\psi(w-hw,k)}-1\right)=\frac{2w}{7k}$$
$$\left(\frac{\psi_j(z-hz,k)}{\psi_j(w-hz,2k)}-1+\frac{\psi_j(w-hw,k)}{\psi_j(w-hz,2k)}-1\right)$$
$$\leq\frac{10}{7}\left(\frac{1}{\psi_j(w-z,k)}-1\right)=\frac{5|w-z|}{7k}$$
$$\left(\frac{1}{\psi_j(z-hz,k)}-1+\frac{1}{\psi_j(w-hw,k)}-1\right)=\frac{2|w-z|}{7k}$$

Therefore, Theorem 2.6's requirements are all met with $r=\frac{3}{7},s=t=\frac{1}{9}$, and $v=\frac{1}{12}$, and h has a fixed point $4\in V$.

3. Application

To support our effort, we have demonstrated an integral-type application. Consider $V=C[0,n],n\in Z^{+}$ to represent the space of continuous mappings with real values defined on the closed interval $[0,n],n\in Z^{+}$. Let

$$w(\beta)=\int_0^{\beta}\phi(\beta,x,w(x))dx \tag{14}$$

is the nonlinear integral equation, where $\beta,x\in[0,n],\forall w\in V$, and $\phi:[0,n]\times[0,n]\times R\to R$ is continuous mapping with real values. Let $A=\{F_n\}_{n\in Z^{+}}$ be the family of semi-norms defined as:

$$F_n(w-z)=\sup_{\beta\in[0,n]}|w(\beta)-z(\beta)|,\forall w,z\in V$$

Consider the countable separating family of fuzzy semi-norms $E=\{\psi_n\}_{n\in Z^{+}}$ defined as:

$$\psi_n(w-z,k)=\frac{k}{F_n(w-z)+k} \tag{15}$$

For $k>0,\forall w,z\in V$, with continuous t-norm $*$ define as: $a*b=ab,\forall a,b\in[0,1]$. It can be easily verified that $E=\{\psi_n\}_{n\in Z^{+}}$ is triangular and V is a fuzzy Fréchet space.

Now, consider the integral Eq. (14), Define the operator $h:V\to V$ by

$$hw(\beta)=\int_0^{\beta}\phi(\beta,x,w(x))dx \tag{16}$$

$\forall w\in V$, and $\exists\lambda,0<\lambda<1$, such that

$$F_n(hw-hz)\leq\lambda M(h,w,z) \tag{17}$$

$\forall F_n\in A$, and $\forall w,z\in V$ where

$$M(h,w,z)=\max\{F_n(w-z),2F_n(w-hw)\} \tag{18}$$

$\forall F_n\in A$, and $\forall w,z\in V$. So, there is only one solution to the integral Eq. (14) in V. Observe that h has a unique fixed point in V if and only if Eq. (14) has a unique solution. Moreover, h is well-defined. we now have to demonstrate that the integral operator h is covered by Theorem(2.4). We have the following two situations, $\forall w,z\in V$

1. If $M(h,w,z)=F_n(w-z)$, in Eq. (18) , from Eqs. (15) and (17) , we get:

$$\left(\frac{1}{\psi_n(hw-hz,k)}-1\right)=\frac{F_n(hw-hz)}{k}$$
$$\leq\lambda\frac{M(h,w,z)}{k}$$
$$=\lambda\frac{F_n(w-z)}{k}$$
$$=\lambda\left(\frac{1}{\psi_n(w-hz,k)}-1\right).$$

This suggests that,

$$\left(\frac{1}{\psi_n(hw-hz,k)}-1\right)\leq\lambda\left(\frac{1}{\psi_n(w-hz,k)}-1\right), \tag{19}$$

$\forall\psi_n\in E,\forall k>0$, and $\forall w,z\in V$ such that $hw\neq hz$. If $hw=hz$ the inequality Eq. (19) holds. With $r=\lambda$ and $s=0$, the integral operator h thus satisfies all Theorem (2.4)'s requirement. Consequently, the uniqueness of fixed point of h ensures that Eq. (14) admits a solution in the space V.

2. If $M(h,w,z)=F_n(w-hw)$, in Eq. (18) , from equations Eqs. (15) and (17), we get:

$$\left(\frac{1}{\psi_n(hw-hz,k)}-1\right)=\frac{F_n(hw-hz)}{k}$$
$$\leq\lambda\frac{M(h,w,z)}{k}$$
$$=\lambda\frac{F_n(w-hw)}{k}$$
$$\leq 2\lambda\frac{F_n(w-hw)}{k}.$$

This suggest that

$$\left(\frac{1}{\psi_n(hw-hz,k)}-1\right)\leq 2\lambda\frac{F_n(w-hw)}{k}, \tag{20}$$

$\forall\psi_n\in E, \forall k>0$,. Using (3) in definition(1.2) and inequality Eq. (15), we may simplify the expression

$$\left(\frac{\psi_n(w-z,k)}{\psi_n(w-hw,k)*\psi_n(z-hw,2k)}-1\right),$$

in this case. For $k>0$, we have:

$$\begin{aligned}&\left(\frac{\psi_n(w-z,k)}{\psi_n(w-hw,k)*\psi_n(z-hw,2k)}-1\right)\\ &\leq\left(\frac{\psi_n(w-z,k)}{\psi_n(w-hw,k)*\psi_n(z-w,k)*\psi_n(w-hw,k)}-1\right)\\ &=\frac{1}{(\psi_n(w-hw,k))^2}-1\\ &=\frac{(k+F_n(w-hw))^2-k^2}{k^2}\\ &=\frac{2F_n(w-hw)}{k}+\left(\frac{F_n(w-hw)}{k}\right)^2\end{aligned}$$

This suggest that

$$\begin{aligned}&\left(\frac{\psi_n(w-z,k)}{\psi_n(w-hw,k)*\psi_n(z-hw,2k)}-1\right)\\ &\leq\frac{2F_n(w-hw)}{k}+\left(\frac{F_n(w-hw)}{k}\right)^2\end{aligned} \tag{21}$$

$\forall\psi_n\in E$, and for $k>0$. From inequalities Eqs. (20) and (21), we have

$$\begin{aligned}&\frac{1}{\psi_n(hw-hz,k)}-1\Big)\\ &\leq\lambda\left(\frac{\psi_n(w-z,k)}{\psi_n(w-hw,k)*\psi_n(z-hw,2k)}-1\right)\end{aligned}$$

$\forall\psi_n\in E$, for $k>0$ and $\forall w,z\in V$ such that $hw\neq hz$. If $hw=hz$, the inequality holds. With $s=\lambda$ and $r=0$, the integral operator h thus satisfies all Theorem (2.4)'s requirement. The mapping h admits a single fixed point, which implies that the nonlinear integral Eq. (14) possesses a solution within the space V.

4. Conclusion

In this study, we established fixed point results within the framework of fuzzy Fréchet spaces by employing the concept of rational type contraction with the triangular property of fuzzy semi-norms. As an application of these results, we investigated a nonlinear integral equation and demonstrated the existence and uniqueness of its solution, thereby highlighting the pratical value of the theoretical findings. The outcomes of this work lay a soild foundation for the further exploration into the role of fixed point principles in fuzzy Fréchet spaces, particularly in analysing the solvability of various functional and integral equations.

References

[1] S. Banach, (1922) *"Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales"* **Fundamenta mathematicae** *3*(1): 133–181. DOI: eudml.org/doc/213289.

[2] A. G. Jasim and Z. Al-Nafie. "Some fixed point theorems in fuzzy Fréchet manifold". In: *Proceeding of the 1st international conference on advanced research in pure and applied science (ICARPAS2021): Third Annual Conference of Al-Muthanna University/College of Science*. ***2398***. 1. AIP Publishing LLC. 2022, 060007. DOI: 10.1063/5.0095582.

[3] A. G. Jasim, A. A. Sangoor, A. S. Mohammed, T. H. Dahess, and A. H. Kamil, (2024) *"Common fixed point theorem in fuzzy Fréchet space"* **Journal of Interdisciplinary Mathematics** *27*(4): 843–847. DOI: 10.47974/JIM-1881.

[4] C. S. Rao, S. R. Kumar, and K. Sarma, (2024) *"Fixed Point Theorems On 4-Dimensional Ball Metric Spaces And Their Applications"* **Journal of Applied Science and Engineering** *27*(11): 3583–3588. DOI: 10.6180/jase.202411_27(11).0014.

[5] R. I. Sabri and B. A. A. H. Ahmed, (2023) *"Best proximity point results in fuzzy normed spaces"* **Science and Technology Indonesia** *8*(2): 298–304. DOI: 10.26554/sti.2023.8.2.298-304.

[6] A. Karlsson, (2024) *"A metric fixed point theorem and some of its applications"* **Geometric and Functional Analysis** *34*(2): 486–511. DOI: 10.1007/s00039-024-00658-x.

[7] R. I. Sabri, (2025) *"N*-Iteration Approach For Approximation Of Fixed Points In Uniformly Convex Banach Space"* **Journal of Applied Science and Engineering** *28*(8): 1671–1678. DOI: 10.6180/jase.202508_28(8).0005.

[8] S. Dhenakaran, E. Naganathan, and C. Ganesamoorthy, (2008) *"Multiple symmetric keys using Banach fixed point theorem"* **Journal of Discrete Mathematical Sciences and Cryptography** *11*(5): 579–587. DOI: 10.1080/09720529.2008.10698210.

[9] L. A. Zadeh, (1965) *"Fuzzy sets"* **Information and control** *8*(3): 338–353. DOI: S001999586590241X.

[10] A. Katsaras, (1984) *"Fuzzy topological vector spaces II"* **Fuzzy sets and systems** *12*(2): 143–154. DOI: 0165011484900344.

[11] I. Sadeqi and F. S. Kia, (2009) *"Fuzzy normed linear space and its topological structure"* **Chaos, Solitons & Fractals *40***(5): 2576–2589. DOI: 10.1016/j.chaos.2007.10.051.

[12] A. G. Jasim and Z. Al-Nafie. "Fréchet spaces via fuzzy structures". In: *Journal of Physics: Conference Series*. ***1818***. 1. IOP Publishing. 2021, 012082. DOI: 10.1088/1742-6596/1818/1/012082.

[13] A. G. Jasim and Z. Al-Nafie. "Fuzzy Fréchet Manifold". In: *Journal of Physics: Conference Series*. ***1818***. 1. IOP Publishing. 2021, 012064. DOI: 10.1088/1742-6596/1818/1/012064.

[14] J. Shamoona, U. R. Saif, Z. Zhiming, and W. Wei, (2020) *"Weakly compatible and quasi-contraction results in fuzzy cone metric spaces with application to the Urysohn type integral equations"* **Advances in Continuous and Discrete Models *2020***(1): DOI: 10.1186/s13662-020-02743-5.

[15] B. Schweizer, A. Sklar, et al., (1960) *"Statistical metric spaces"* **Pacific J. Math *10***(1): 313–334. DOI: 10.2140/pjm.1960.10.313.