Backstepping Method For Stabilizing Fuzzy Parabolic Equation Using Difference Method

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In this work, a set of laws has been proposed as a group of feedback control stabilizing a boundary law for a linear fuzzy reaction-advection-diffusion equation. Stabilization is achieved by designing coordinate transformations that form recursive relationships; by using the fuzzy finite difference method, we can convert coordinates into other coordinates. This design process is unlimited to any specific kinds of boundary actuation and can handle systems with an arbitrarily finite number of eigenvalues for the unstable open-loop system. We noticed that there is another problem when converting coordinates, which is that the equation includes lower and upper functions, so we wrote the equations in the form of matrices and then converted them into ordinary differential equations. The problem of feedback, which becomes increasingly unbounded as the grid gets infinitely fine, is solved by carefully selecting the target system to which the original system is transformed. Then we stabilize the closed loop system and the regularity of control and solutions to the fuzzy reaction-advection-diffusion equation.

Keywords: Fuzzy reaction-advection-diffusion equation, Fuzzy Backstepping control method, Fuzzy finite difference method, Fuzzy Volterra transformation.

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1. Introduction

Since 1965, when Zadeh introduced for the first time the theory of fuzzy sets as a logic which extends the classical logic set theory, scientists have been interested in this theory, as intended, the interest is dissected by some of them towards fuzzy dynamical systems, because of their wide importance in scientific fields, such as fuzzy reaction, advection, heat, and fuzzy wave equations, control, stability, and other science areas (see for more details [1, 2]. Hence, we can consider fuzzy logic has garnered significant interest and its theory has become of researcher's importance in all areas of economic and scientific life starts from the end of the last century. Moreover, since most of real-world applied sciences and engineering systems, such as thermal diffusion procedures and chemical engineering, can involve spatial and temporal evolution, which can be modeled in general using partial differential equations (PDEs) and by reaction advection diffusion equation (RADE) that can cover some natural phenomena like heat transfer, diffusion, fluid flow, and fusion plasma transport.

In addition to the above, control theory for PDEs is an active research field, and hence directed the main work of this paper [3–7] and so for linear parabolic PDEs involving systems of RADE, various control strategies have been well investigated in the literature. In recent years, the so-called backstepping approach has provided a surprising systematic and boundary-stabilizing controller synthesis for these classes of PDEs [8–10].

Belman-Flores (2022), used the fuzzy control principle in refrigeration and air conditioning systems [11], and studied of fuzzy control methods of robotic manipulators using data-driven PID techniques is presented see [12]. In [13] introduced a new combination called Fuzzy Backstepping Control is established in this work to eliminate the influence of error and phase difference phenomena, which is considered a new contribution to. in [14] proposes an adaptive fuzzy output feedback control approach for quadrotor unmanned aerial vehicles with stochastic disturbances, besides which we consider the unmeasurable states and unknown nonlinear functions. Motivated by the model utilized in engineering applications, the authors provide an approach for global stabilization of a larger class of linear fuzzy parabolic PDEs.

The main objective of this article, as well as the central principle of the proposed fuzzy PDE backstepping approach, is to introduce a fuzzy Volterra integral transformation (fuzzy VIT) that is capable to force an unstable fuzzy PDE system to a well-chosen exponentially stable target fuzzy PDE system. Kernel equations are used to produce the kernel of a fuzzy VIT, and fuzzy backstepping synthesis offers a framework for dealing with issues involving control functions acting on boundary conditions.

The main issue with unstable linear fuzzy parabolic PDE systems is the system of fuzzy into the target system being transformed by coordinate transformation. Standard backstepping routes can result in finite kernels, while proper selection of the target system leads to a bounded kernel and continuous solutions to the controlled problem. Therefore, it is crucial to choose the right system for accurate transformation. The fundamental problem in unstable linear equivalent partial differential equation systems is the target system to which the original system is transformed by coordinates. Therefore, in this paper we take the stable fuzzy PDE, to find the kernel and from it find the control function.

The structure of this paper is as follows; In the introduction, we introduce a brief overview of the fuzzy set and fuzzy theory, as well as, considering the fuzzy differential equation. In the second section, we mentioned the most important definitions that we need for this work. In the third section, we discussed the fuzzy RADE in its general form and also its special case. In the fourth section, we mentioned the analytical solution of a special example of fuzzy RADE with two different cases of the Hukuhara's derivation. In the fifth section, we applied the difference method to the equation and derive the convergent kernel. Finally, two different examples are given and applying the obtained theoretical results for these equations.

2. Basic methods

Backstepping method and difference method are two methods used in the stability of systems of differential equations. First, we will start with the backstepping approach for fuzzy RADE. The stable fuzzy system can be obtained if the fuzzy differences method is used based on the following the steps as the main scheme:

Step 1: Divide the space domain into N-equally spaced node points, where $N \in \mathbb{N}$ and using the finite difference approximations for the time derivatives.

Step 2: Defuzzification of the fuzzy differential equation based on Hukukara differentiability and level wise crisp sets related to fuzzy set.

Step 3: Rewrite the system in matrix form of ordinary differential equations.

Step 4: Consider the finite-dimensional backstepping approach of coordinate transformation.

Step 5: By performing multiple calculations, we obtain the recurrence relationship through which we can obtain the kernel function that can stabilize the unstable system after substituting it into the control equation.

2.1. Backstepping Approach for Fuzzy Reaction-Advection-Diffusion Equation

The fuzzy controller is defined and used based on the fact that most problems in various applied sciences and also in real-life applications include inaccurate, ambiguous or vague information or parameters. Therefore, the decision which must be taken when solving these problems cannot be determined accurately, but rather it will be within a range of values varying between 0 and 1 that are labeled as the α-levels, $\alpha \in [0,1]$. This will depend on the extent of the property's realization, and this will also be a generalization of non-fuzzy logic. Moreover, in the backstepping method, the fuzzy controller will appear, which will be within a certain range of values ranging between 0 and 1. The method followed in this research is through the combination between the theory of intervals and the interpretation of fuzzy functions and variables through the α-level sets.

Consider the general 1-dimensional fuzzy parabolic PDE is:

$$\begin{aligned} \tilde{u}_t &= \gamma \tilde{u}_{xx} + \tilde{\rho}\tilde{u}_{tx} + \tilde{\beta}\tilde{u}_x \\ &\quad + \tilde{\lambda}(x)\tilde{u}(x,t) + \tilde{\mathcal{G}}(x)\tilde{u}(0,t), \quad x \in [0,1], t > 0. \end{aligned} \tag{1}$$

with initial and boundary conditions

$$\tilde{u}(0,t) = \tilde{0}, \tag{2}$$

$$\tilde{u}(1,t) = \tilde{U}(t) \tag{3}$$

$$\tilde{u}(x,0) = \tilde{g}(x) \tag{4}$$

where $\tilde{u}_t(x,t), \tilde{u}_x(x,t), \tilde{u}_{tx}(x,t)$, and $\tilde{u}_{xx}(x,t)$ are fuzzy partial derivatives in the sense of Hukuhara partial differentiability [15, 16], γ is a diffusion constant, $\gamma\tilde{u}_{xx}(x,t)$ is the diffusion term with $\tilde{\lambda}(x)\tilde{u}(x,t)$ is the reaction term, $\tilde{\beta}$ is the diffusion velocities in the x-direction [17–19], $\tilde{U}(t)$ is the fuzzy control, and $\tilde{\lambda}(x)$ is a fuzzy function defined as it is defined in Definitions Eqs. (4) and (5).

Now, let's take the special case of the parabolic PDE, which is of the form:

$$\tilde{u}_t = \gamma\tilde{u}_{xx} + \beta\tilde{u}_x + \tilde{\lambda}(x)\tilde{u}(x,t) \tag{5}$$

this equation is called the fuzzy RADE, and with initial and boundary conditions as in Eqs. (2) to (4).

Because Eq. (5) is unstable due to the introduction of the fuzzy eigenvalues $\tilde{\lambda}(x)$. Now using coordinate fuzzy backstepping approach fuzzy VIT equation (TVIE) as:

$$\tilde{W}(x,t) = \tilde{u}(x,t) - \int_0^x \tilde{k}(x,\xi)\tilde{u}(\xi,t)d\xi, \quad x \in \mathbb{R}, t \geq 0 \tag{6}$$

with feedback control

$$\tilde{u}(1,t) = \int_0^1 \tilde{k}(1,\xi)\tilde{u}(\xi,t)d\xi. \tag{7}$$

Then system Eqs. (2) to (4) and Eq. (5) can be transformed into the target following system:

$$\tilde{W}_t = \gamma\tilde{W}_{xx} + \beta\tilde{W}_x - \tilde{C}\tilde{W}(x,t), \quad x \in \mathbb{R}, t \geq 0 \tag{8}$$

$$\tilde{W}(0,t) = \tilde{0} \tag{9}$$

$$\tilde{W}(1,t) = \tilde{0} \tag{10}$$

that are exponentially stable for $\tilde{C} > \frac{-\gamma\pi^2-\beta^2}{4\gamma}$.

In this research we need to the most important definitions that are important as follows:

Definition 1, [20, 21]: The crisp set which contains the set of all elements of X with membership function $\mu_{\tilde{A}}(x) \geq \alpha$, is called the α-level set can be written in mathematical symbols to be described as:

$A_\alpha = \{(x, \mu_{\tilde{A}}(x) \geq \alpha) : \alpha \in [0,1]\}$.

When concerned with fuzzy PDEs, fuzzy numbers (FNs) play an important role in their definitions, in which different types of FNs exist in the literature. In this work certain type of FNs will be considered, which is the trapezoidal FN, as in the following defined:

Definition 2, [22]: A FN with the membership function:

$$\mu(x,\sigma,\mathfrak{b},\mathfrak{c},\mathfrak{d}) = \begin{cases} 1 & \mathfrak{h} \leq x \leq \mathfrak{c} \\ \frac{x-\sigma}{\mathfrak{h}-\sigma} & \sigma \leq x \leq \mathfrak{h} \\ \frac{\mathfrak{d}-x}{\mathfrak{d}-\mathfrak{c}} & \mathfrak{c} \leq x \leq \mathfrak{d} \\ 0 & x > 0 \end{cases}$$

is called a trapezoidal FN with the base $[\sigma, \mathfrak{d}]$ and vertices at $x = \mathfrak{d}, x = c$ where $\sigma, \mathfrak{h}, \mathfrak{c}$, and $\mathfrak{d} \in \mathbb{R}$, and $\sigma < \mathfrak{h} < \mathfrak{c} < \mathfrak{d}$.

Definition 3, [23]: A trapezoidal FN $\tilde{\mathrm{N}}$ in parametric form is a pair $[\underline{N}(\alpha), \bar{N}(\alpha)], 0 \leq \alpha \leq 1$, which satisfies the axioms below:

1. $\underline{N}(\alpha)$ is an increasing, bounded and monotonic function on $[\sigma, \mathfrak{h}]$.
2. $\bar{N}(\alpha)$ is a decreasing, bounded and monotonic function on $[c, \mathfrak{d}]$.
3. $\underline{N}(\alpha) \leq \bar{N}(\alpha)$, for all $0 \leq \alpha < 1$.
4. $[\underline{N}(\alpha), \bar{N}(\alpha)] = [\mathfrak{h}, \mathfrak{c}]$, if $\alpha = 1$.

Definition 4, [24]: Fuzzifying function (FF) $\tilde{\mathfrak{g}} : X \to Y$ is the ordinary mapping of X into fuzzy power set $\tilde{P}(Y), \tilde{\mathfrak{g}} : X \to \tilde{\mathfrak{g}}(x)$, that is to say, the FF is a mapping from domain to fuzzy set of range, FF and the fuzzy relation coincides with each in the mathematical manner. So, to speak, FF can be interpreted as fuzzy relation $\tilde{\mathfrak{R}}$ defined as follows:

$\forall(x,y) \in X \times Y, \mu_{\mathfrak{g}(x)}(y) = \mu_{\mathfrak{R}}(x,y)$, where $y = \mathfrak{g}(x)$.

Definition 5, [24] : Let $\tilde{\mathfrak{g}}$ be a FF from $[\sigma, \mathfrak{h}] \subseteq \mathbb{R}$ to $\mathbb{R}$, such that, $\forall x \in [\sigma, \mathfrak{h}]\tilde{\mathfrak{g}}(x)$ is a FN, i.e., a piecewise continuous convex normalized fuzzy set on $\mathbb{R}$. For all $\alpha \in (0,1]$, the equation $\mu_{\tilde{g}(x)}(y) = \alpha$, with x and α as parameters is assumed to have two and only two continuous solutions $y = \bar{\mathfrak{g}}(x;\alpha)$ and $y = \underline{\mathfrak{g}}(x;\alpha)$, for $\alpha \neq 1$ and only one solution $y = \mathfrak{g}(x)$, for $\alpha = 1$, which is also continuous. The functions $\bar{\mathfrak{g}}(x;\alpha)$ and $\underline{g}(x;\alpha)$ are defined such that:

$\bar{\mathfrak{g}}(x;\dot{\alpha}) \geq \bar{\mathfrak{g}}(x;\alpha) \geq \mathfrak{g}(x) \geq \underline{\mathfrak{g}}(x;\alpha) \geq \underline{\mathfrak{g}}(x;\dot{\alpha}), \forall \alpha, \dot{\alpha}$ with $\alpha \geq \dot{\alpha}$

These functions are called α-level curves of $\tilde{\mathfrak{g}}(x)$.

Definition 6, [24, 25]: Let $\tilde{A}, \tilde{B} \in \mathbb{R}_g$ and if there exist $\tilde{P} \in \mathbb{R}_g$, such that $\tilde{B} = \tilde{A} + \tilde{P}$, then $\tilde{P}$ is called the Hukuhara difference of $\tilde{B}$ and $\tilde{A}$, which is denoted by $\tilde{B} \ominus \tilde{A}$.

As in FNs, the α-level representation of fuzzy-real valued function $\tilde{\mathfrak{g}} : [\sigma, \mathfrak{h}] \to \mathbb{R}_{\mathfrak{g}}$ is expressed by the closed interval, $\tilde{\mathfrak{g}}(x;\alpha) = [\underline{\mathfrak{g}}(x;\alpha), \bar{\mathfrak{g}}(x;\alpha)], x \in [\sigma, \mathfrak{h}], \alpha \in [0,1]$.

The generalized Hukuhara differentiation, which is a core concept that extends Hukuhara difference and differentiability, is almost the generic form of fuzzy differentiation for interval-valued functions. Hukuhara invented the Hukuhara derivative, or H-derivative, in 1976, which later served as the foundation for research into fuzzy differential equations. Hukuhara derivative is possibly considered a generalization of the nonfuzzy or crisp derivative, as it is seen in the next definitions in [26, 27]:

Definition 7: Suppose that $\tilde{\mathfrak{g}} : D \subseteq \mathbb{R} \to \mathbb{R}$ is the fuzzy-valued function of two variables are rule that assigns to each order pair of real numbers, (x, t) in a set D, a unique FN denoted by $\tilde{\mathfrak{g}}(x, t)$. The set D is the domain of $\tilde{\mathfrak{g}}(x, t)$ and $(\tilde{\mathfrak{g}}(x, t))\alpha = [\underline{\mathfrak{g}}(x, t; \alpha), \overline{\mathfrak{g}}(x, t; \alpha)]$. If there exist partial derivatives of $\underline{\mathfrak{g}}(x; \alpha)$ and $\overline{\mathfrak{g}}(x; \alpha)$ with respect to $x \in$ D and the interval $\left[\frac{\partial \underline{\mathfrak{g}}(x;\alpha)}{\partial x}, \frac{\partial \overline{\mathfrak{g}}(x;\alpha)}{\partial x}\right]$, for $(x, \mathrm{t}) \in \mathrm{D}, \alpha \in [0, 1]$ defines the α-leval set of a FN , then $\tilde{\mathfrak{g}}(x, \mathrm{t})$ is called differential and written as:

$(\partial\tilde{\mathfrak{g}}(x, \mathrm{t}))_\alpha = \left[\frac{\partial \underline{\mathfrak{g}}(x,t;\alpha)}{\partial x}, \frac{\partial \overline{\mathfrak{g}}(x,t;\alpha)}{\partial x}\right]$.

New our focus in this part is to analyse fuzzy ARDE with p- gH-differentiability based on α level sets which will be carried through assuming, $u^2(\tilde{x}, t) = [-u(x, t; \alpha), {}^-u(x, t; \alpha)]$, then substituting in the governing PDE given by Eq. (5), we get:

$$\frac{\partial}{\partial t}\underline{u}(x, t; \alpha) =$$

$$\min\left\{\gamma\frac{\partial^2}{\partial x^2}\underline{u}(x, t; \alpha) + \beta\frac{\partial}{\partial x}\underline{u}(x, t; \alpha) + \underline{\lambda u}(x, t; \alpha)\right.$$
$$\left.\gamma\frac{\partial^2}{\partial x^2}\bar{u}(x, t; \alpha) + \beta\frac{\partial}{\partial x}\bar{u}(x, t; \alpha) + \bar{\lambda}\bar{u}(x, t; \alpha)\right\}$$

$$\frac{\partial}{\partial t}\bar{u}(x, t; \alpha) =$$

$$\max\left\{\gamma\frac{\partial^2}{\partial x^2}\underline{u}(x, t; \alpha) + \beta\frac{\partial}{\partial x}\underline{u}(x, t; \alpha) + \underline{\lambda u}(x, t; \alpha),\right.$$
$$\left.\gamma\frac{\partial^2}{\partial x^2}\bar{u}(x, t; \alpha) + \beta\frac{\partial}{\partial x}\bar{u}(x, t; \alpha) + \bar{\lambda}, \bar{u}(x, t; \alpha)\right\}$$

Therefore, two cases arise from above Equations, as follows:

Case (i): If $\underline{u}_t(x, t; \alpha) \leq \bar{u}_t(x, t; \alpha)$ then the possible PDEs resulted from system Eqs. (2) to (5) are:

$\underline{u}_t = \gamma\underline{u}_{xx} + \beta\underline{u}_x + \underline{\lambda u}(x, t; \alpha)$

$\bar{u}_t = \gamma\bar{u}_{xx} + \beta\bar{u}_x + \bar{\lambda}\bar{u}(x, t; \alpha)$

with initial and boundary conditions for lower and upper respectively

$$\underline{u}(0, t; \alpha) = \underline{0}, \tag{11}$$

$$\begin{aligned} \underline{u}(1, t; \alpha) &= \underline{U}(t; \alpha) \\ \underline{u}(x, 0; \alpha) &= \underline{g}(x; \alpha) \end{aligned} \tag{12}$$

and

$$\bar{u}(0, t; \alpha) = \bar{0}, \tag{13}$$

$$\begin{aligned} \bar{u}(1, t; \alpha) &= \bar{U}(t; \alpha) \\ \bar{u}(x, 0) &= \bar{g}(x; \alpha) \end{aligned} \tag{14}$$

Case (ii): If $\underline{u}_t(x, t; \alpha) > \bar{u}_t(x, t; \alpha)$, then the possible PDEs resulted from system Eqs. (2) to (5) are:

$\underline{u}_t(x, t; \alpha) = \gamma\bar{u}_{xx} + \beta\bar{u}_x + \bar{\lambda}(x)\bar{u}(x, t; \alpha)$

$\bar{u}_t(x, t; \alpha) = \gamma\underline{u}_{xx} + \beta\underline{u}_x + \underline{\lambda}(x)\underline{u}(x, t; \alpha)$

with the same initial and boundary conditions given by Eqs. (11) to (14).

In the next section, we will utilize the fuzzy finite difference technique (fuzzy FDM) to discover the kernel $\tilde{k}(x, \xi)$ for the purpose of studying the possibility of stabilization of the fuzzy RADE.

2.2. Fuzzy Finite Difference Method with Fuzzy Backstepping Coordinate

In this section we use finite difference method, this approach may be illustrated using three steps:

Step 1: The followed approach starts by dividing the space domain into N-points, and by using the finite difference approximations for the time derivatives [28], we obtain:

$$\tilde{u}_0 = \tilde{0}$$

$$\tilde{u}_i = \gamma\frac{\tilde{u}_{i+1} - 2\tilde{u}_i + \tilde{u}_{i-1}}{h^2}$$

$$+\beta\frac{\tilde{u}_{i+1} - \tilde{u}_i}{h} + \tilde{\lambda}_i(x)\tilde{u}_i, \quad i = 1, 2 \ldots, n.$$

$$\tilde{u}_{n+1} = \varphi_n(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n),$$

Where $n \in \mathbb{N}, h = \frac{1}{(N+1)}, \tilde{u}_i = \tilde{u}(ih, t), \tilde{\lambda}_i = \tilde{\lambda}(ih)$, for all $i = 0, 1, \ldots, N+1, \tilde{u}_0$ is the first boundary condition, $\tilde{u}_{N+1}$ is the evaluated control function, and in terms of $\alpha-$ levels as non-fuzzy parametric equations, we have two cases as in the next:

Step 2: Fuzzifying the fuzzy differential equation based on Hukuhara derivative cases.

For Case (i), obtain:

$$\underline{u}_0 = \underline{0} \tag{15}$$

$$\dot{\underline{u}}_i(x, t; \alpha) = \gamma\frac{\underline{u}_{i+1} - 2\bar{u}_i + \underline{u}_{i+1}}{h^2} + \beta\frac{\underline{u}_{i+1} - \bar{u}_i}{h} + \underline{\lambda}_i(x; \alpha)\underline{u}_i(x, t; \alpha) \tag{16}$$

$$\underline{u}_{n+1} = \underline{\varphi}_n(\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n), \tag{17}$$

and

$$\bar{u}_0 = \bar{0}, \tag{18}$$

$$\bar{u}_i(x, t; \alpha) = \gamma\frac{\bar{u}_{i+1} - 2\underline{u}_i + \bar{u}_{i-1}}{h^2} + \beta\frac{\bar{u}_{i+1} - \underline{u}_i}{h} + \bar{\lambda}_i(x; \alpha)\bar{u}_i(x, t; \alpha), \tag{19}$$

$$\bar{u}_{n+1} = \bar{\varphi}_n(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n), \tag{20}$$

Step 3: We can rewrite the system as a matrix differential equation of ordinary differential equation as:

$$v_0 = 0, \tag{21}$$

$$\dot{v}_i = (\gamma + h\beta)v_{i+1} - (2\gamma + h\beta)I^{'}v_i + \gamma v_{i-1} + h^2\lambda_i(x;\alpha)v_i, \tag{22}$$

$$v_{n+1} = \varphi_i\left(v_1, v_2, \ldots, v_{i-1}\right), \tag{23}$$

Where $n \in \mathbb{N}, i = 0, 1, \ldots n+1$. With v_{n+1} as control, where $v_i = \begin{bmatrix} \underline{u}_i(x,t;\alpha) \\ \bar{u}_i(x,t;\alpha) \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \varphi_i = \begin{bmatrix} \underline{\varphi}_i\left(\underline{u}_1, \ldots, \underline{u}_n\right) & 0 \\ 0 & \bar{\varphi}_i\left(\bar{u}_1, \ldots, \bar{u}_n\right) \end{bmatrix},$

$\lambda_i = \begin{bmatrix} \underline{\lambda}_i & 0 \\ 0 & \bar{\lambda}_i \end{bmatrix},$

and

$I^{'} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$

Noting that, Eq. (22) is unstable because of the existence of the eigenvalues for all λ_i, for all i, and applying fuzzy FDM for TVIE by into ODE as follows:

$$\omega_i(x,t;\alpha) = v_i(x,t;\alpha) - \int_0^x \tilde{I}k_i(x,\xi;\alpha)v_i(\xi,t;\alpha)d\xi \tag{24}$$

Where $k_i(x,\xi)$ is a gain kernel function, in addition to feedback control defined as:

$$v_i(1,t;\alpha) = \int_0^1 I^{'}k_i(1,\xi;\alpha)v_i(\xi,t;\alpha)d\xi \tag{25}$$

Where $k_i(x,y;\alpha) = \begin{bmatrix} \underline{k}_i & 0 \\ 0 & \bar{k}_i \end{bmatrix}$

Step 4: We start the methodology with a finite-dimensional fuzzy backstepping transformations:

$$\omega_0 = v_0 = \begin{bmatrix} \underline{0} \\ \bar{0} \end{bmatrix} \tag{26}$$

$$\omega_i = v_i - I^{'}\varphi_{i-1}\left(v_1, v_2, \ldots, v_{i-1}\right) \tag{27}$$

$$\omega_{n+1} = \begin{bmatrix} \underline{0} \\ \bar{0} \end{bmatrix}, \tag{28}$$

Now convert Eqs. (7) to (9) into new target system by finite different method and matrix differential equation obtain:

$$\omega_0 = \begin{bmatrix} \underline{0} \\ \bar{0} \end{bmatrix} \tag{29}$$

$$\dot{\omega}_i = (\gamma + h\beta)\omega_{i+1} - (2\gamma + h\beta)I^{'}\omega_i + \gamma\omega_{i-1} - h^2 I^{'}C(x;\alpha)\omega_i \tag{30}$$

$$\omega_{n+1}(1,t) = \begin{bmatrix} \underline{0} \\ \bar{0} \end{bmatrix} \tag{31}$$

Now, from system Eqs. (21) to (23) and system Eqs. (29) to (31), and upon using system transformation to find the function is φ_i, and hence the backstepping coordinate transformation by substituting Eqs. (26) to (28) into Eqs. (29) to (31), we get:

$$\dot{v}_i - I^{'}\sum_{j=1}^{i-1}\frac{\partial\varphi_{i-1}}{\partial v_j}\dot{v}_j\left(v_1, v_2, \ldots, v_{i-1}\right)$$
$$= -(2\gamma + h\beta)\hat{I}\left(v_i - \hat{I}\varphi_{i-1}\right)$$
$$+\gamma\left(v_{i-1} - \dot{I}\varphi_{i-2}\right) - h^2 I^{'}C\left(v_i - \dot{I}\varphi_{i-1}\right)$$

Since $\dot{v}_i = (\gamma + h\beta)v_{i+1} - (2\gamma + h\beta)\hat{I}v_i + \gamma v_{i-1} + h^2\lambda_i(x;\alpha)v_i,$

$v_{n+1} = \varphi_i\left(v_1, v_2, \ldots, v_{i-1}\right)$

then

$$(\gamma + h\beta)\varphi_i - (2\gamma + h\beta)I^{'}v_i + \gamma v_{i-1}$$
$$+h^2\lambda_i(x;\alpha)v_i - I\sum_{j=1}^{i-1}\frac{\partial\varphi_{i-1}}{\partial v_j}\dot{v}_j\left(v_1, v_2, \ldots, v_{i-1}\right)$$
$$= -(2\gamma + h\beta)I^{'}\left(v_i - I^{'}\varphi_{i-1}\right)$$
$$+\gamma\left(v_{i-1} - I^{'}\varphi_{i-2}\right) - h^2 I^{'}\left(v_i - I\varphi_{i-1}\right)$$

By solving the above equations and applying Eqs. (21) to (23) we can find $\boldsymbol{\varphi_i}$ as follows:

$$\varphi_i = \frac{I^{'}}{\gamma + h\beta} \times \begin{pmatrix} (2\gamma + h\beta)\varphi_{i-1} + h^2 I^{'}CI^{'}\varphi_{i-1} - \gamma I^{'}\varphi_{i-2} - \\ h^2\left(I^{'}C + \lambda_i\right)v_i + I\sum_{j=1}^{i-1}\frac{\partial\varphi_{i-1}}{\partial v_j}\dot{v}_j\left(v_1, v_2, \ldots, v_{j-1}\right) \end{pmatrix}$$

for i=1,2, …, n , and from $\varphi_0 = \varphi_{-1} = 0$, we have $\varphi_1 = \frac{-Ih^2}{\gamma + h\beta}\left(I^{'}C + \lambda_1\right)v_1.$

We can write the φ_i by linear summation as:

$\varphi_i = \sum_{j=1}^{i} k_{i,j}v_j$

through simple calculations, we can derive a general recursive relationship:

$$k_{i,1} = \frac{-I^{'}h^2}{\gamma + h\beta}\left(I^{'}C + \lambda_1\right)k_{i-1,1} + \frac{-ih^2}{\gamma + h\beta}\left(k_{i-1,2} - k_{i-2,1}\right) \tag{32}$$

$$k_{i,j} = \frac{-I^{'}h^2}{\gamma + h\beta}\left(I^{'}C + \lambda_1\right)k_{i-1,j} + k_{i-1,j-1} + \frac{I^{'}h^2}{\gamma + h\beta}\left(k_{i-1,j+1} - k_{i-2,j}\right) \tag{33}$$

$$k_{i,i-1} = \frac{I^{'}h^2}{\gamma + h\beta}\left(I^{'}c + \lambda_{i-1}\right)k_{i-1,i-1} + k_{i-1,i-2} \tag{34}$$

$$k_{i,i} = k_{i-1,i-1} - \frac{+I^{'}h^2}{\gamma + h\beta}\left(I^{'}c + \lambda_i\right) \tag{35}$$

for i=1,2,3,4 with initial conditions:

$$k_{1,1} = \frac{-I^{'}h^2}{\gamma + h\beta}\left(I^{'}C + \lambda_1\right) \tag{36}$$

$$k_{2,1}=\frac{-I'}{\gamma+h\beta}\left(-\frac{h^4(2\alpha+h\beta)}{(\alpha+h\beta)}I'\left(I'C+\lambda_1\right)(I-I')-\frac{I'h^4}{\alpha+h\beta}\left(I'CI'\left(I'C+\lambda_1\right)+\left(I'C+\lambda_1\right)\lambda_1\right)\right) \tag{37}$$

$$k_{2,2}=-\frac{I'h^2}{\gamma+h\beta}\left(I'C+\lambda_1\right)-\frac{I'h^2}{\gamma+h\beta}\left(I'C+\lambda_2\right) \tag{38}$$

$$k_{3,3}=\frac{I'h^2}{\gamma+h\beta}\left(I'C+\lambda_1\right)-\frac{I'h^2}{\gamma+h\beta}\left(I'C+\lambda_2\right)-\frac{I'h^2}{\gamma+h\beta}\left(I'C+\lambda_3\right) \tag{39}$$

and,

$$k_{4,3}=\frac{-I'h^2}{(\gamma+h\beta)}\left(I'C+\lambda_3\right)k_{3,3}+k_{3,2} \tag{40}$$

$$k_{4,4}=\frac{-I'h^2}{(\gamma+h\beta)}\left(I'C+\lambda_4\right)+k_{3,3} \tag{41}$$

Let's take $\lambda_i(x)=\lambda_i$ is a TFN defined in Definitions 2 and 3, and letting $\tilde{\lambda}_i=[\underline{\lambda},\bar{\lambda}],\tilde{C}=[\underline{C},\bar{C}]$, then, we can solve Eqs. (32-41) is a similar manner give in [29–31], getting:

$$\kappa_{i,i-j}=-I'\binom{i}{j+1}N_n^{j+1}-(i-j)I'\sum_{I=1}^{\left[\frac{1}{2}\right]}\frac{1}{l}\binom{j-l}{l-1}\binom{i-l}{j-2l}N_n^{j-2l+1}E_n^l, \tag{42}$$

for $i=1,2,\ldots,n, j=i-1,i-2,\ldots,1,0$ where $N_n=\frac{h^2}{\gamma+h\beta}\begin{bmatrix}\underline{\lambda} & \bar{C}\\ \underline{C} & \bar{\lambda}\end{bmatrix}$ $E_n=\frac{\gamma}{\gamma+h\beta}$, and $\begin{bmatrix} i\\ j+1\end{bmatrix}$ refer to the computation of I to kernel $j+1$,

Regarding the infinite-dimensional system (5) with (2) and (3), the linearity of the control law $\varphi_i=\sum_{j=1}^{i}k_{i,j}v_j$ offers a stabilizing the boundary feedback control of the form $U(t;\alpha)=\varphi_i(v)=\int_0^1 Ik_{i,j-1}(x)v_j(x,t;\alpha)dx$

Step 5: By performing multiple calculations, we obtain the recurrence relationship through which we can obtain the kernel function that makes the unstable system stable after substituting it in the control equation.

The next theoretical results are given in order to be used later for the purpose of proving the stability of Eq. (42).

Theorem 1, [29–31]: The sequence $\left\{(n+1)k_{n,j}\right\}_{j=1}^{n}$ remains uniformly bounded in n and j as $n\to\infty$.

Lemma 1: The components of the matrix sequence $\left\{k_{i,j}\right\}$ defined in Eqs. (32) to (41) satisfy

$$\left|k_{i,i-j}\right|\le I\binom{i}{j+1}N_n^{j+1}+(i-j)I'\sum_{I=1}^{\left[\frac{j}{2}\right]}\frac{1}{I}\binom{j-l}{l-1}\binom{i-l}{j-2l}N_n^{j-2l+1}E_n^l \tag{43}$$

Where $\tilde{\lambda}=\max_{x\in[0,1]}|\tilde{\lambda}(x)|$

Proof: $|k_{1,1}|\le\frac{ih^2}{\gamma+h\beta}\begin{bmatrix}\underline{\lambda} & \bar{C}\\ \underline{C} & \bar{\lambda}\end{bmatrix}=N_n$

$|k_{2,1}|\le\frac{I'}{\gamma+h\beta}\left(-\frac{h^4(2\gamma+h\beta)}{(\gamma+h\beta)}I'\begin{bmatrix}\underline{\lambda} & \bar{C}\\ \underline{C} & \bar{\lambda}\end{bmatrix}(I-I')+\frac{h^4}{\alpha+h\beta}\left(CI'\begin{bmatrix}\underline{\lambda} & \bar{C}\\ \underline{C} & \bar{\lambda}\end{bmatrix}+\hat{I}\begin{bmatrix}\underline{\lambda} & \bar{\bar{C}}\\ \underline{C} & \bar{\lambda}\end{bmatrix}\lambda_1\right)=\frac{I'h^2(2\gamma+\beta)}{\gamma+h\beta}N_n+h^2\left(CN_n+N_n\lambda_1\right)\right.$

$|k_{2,2}|\le 2N_n^2$

$\vdots$

$|k_{3,3}|\le 3N_n$

Then let $i=j$ from Eqs. (34) and (35) we get:

$|k_{i,i}|\le iN_n$

$|k_{i,i-1}|\le\frac{Ih^2(2\gamma+\beta)}{\gamma+h\beta}N_n+h^2\left(CN_n+N_n\lambda_1\right)+\ldots$

Then inequality Eq. (43) of lemma 1 have been proved.

Lemma 2, [29–31]: The mapping $\check{k}:[0,1]\to L_\infty(0,1)$ given by $x\to\check{k}(x,.)$ is weakly continues.

Lemma 3, [29–31]: Suppose that two functions $\mathcal{W}(x;\alpha)\in L_\infty(0,1)$ and $\mathcal{V}(x;\alpha)\in L_\infty(0,1)$ satisfying the following relationship, $\mathcal{W}(\mathrm{x},\mathrm{t};\alpha)=$

$\mathcal{V}(\mathrm{x},\mathrm{t};\alpha)-\int_0^x\check{k}(x,\xi;\alpha)\mathcal{V}(\xi;\alpha)d\quad x[0,1]$

where $\check{k}\in C_{\mathcal{W}}\left([0,1];L_\infty(0,1)]\right)$.

There exist positive constants m and M , depending on $\check{k}$, such that $m\|\mathcal{W}\|_\infty\le\|\mathcal{V}\|_\infty\le M\|\mathcal{W}\|_\infty$ and $m\|\mathcal{W}\|_2\|\mathcal{V}\|_2M\|\mathcal{W}\|_2$.

Now from Theorem 1 and Lemmas 1–3, the infinite-dimensional coordinate transformation resulted with a specific boundary condition could be found and then used to prove the next theorem, with is exponentially stable in the L_2 (0,1) space and maximized with decay rate $\tilde{C}$.

Theorem 2: For any $\lambda(x)\in L_\infty(0,1)$ and $\gamma,\beta>0$ there exist a function $k\in L_\infty(0,1)$ such that for any $\mathcal{V}^0\in L_\infty(0,1)$ the unique classical solution $\mathcal{V}(x,t)\in C^1\left((0,\infty);C^2(0,1)\right)$ of system Eqs. (5), (2), (3) and (42) is stable exponentially in the space $L_2(0,1)$ and maximum with decay rate C. The precise statements of stability properties are the following:

There exists a constant M, which is positive so that for all t>0, $\|\mathcal{V}(t)\|_2\le M\left\|\mathcal{V}^0\right\|_2e^{-Ct}$ and $\max_{x\in[0,1]}|\mathcal{V}(x,t)|\le M\sup_{x\in[0,1]}\left|\mathcal{V}^0(x)\right|e^{-Ct}$.

Where $\lambda(x)$ and C are 2×2 matrix of lower and upper functions of TFNs and $\mathcal{V}(x,t)$ vector of function.

Now applying the fuzzy backstepping method for case (ii) of the fuzzy RDE, which is found previously in Section 4, that will mention it again as in case (ii) as follows:

Case (ii) : If $\underline{u}_t \geq \bar{u}_t$ then the possible RADE resulted from the system (2-5) are:

$$\underline{u}_t(x,t;\alpha) = \gamma\bar{u}_{xx} + \beta\bar{u}_x + \bar{\lambda}(x)\bar{u}(x,t;\alpha) \tag{44}$$

$$\bar{u}_t(x,t;\alpha) = \gamma\underline{u}_{xx} + \beta\underline{u}_x + \underline{\lambda}(x)\underline{u}(x,t;\alpha) \tag{45}$$

with the same initial and boundary conditions given by Eqs. (11) to (14).

By using FDM to Eqs. (44) and (45), and by we get:

$$v_0 = I'\begin{bmatrix}0\\ \overline{0}\end{bmatrix} \tag{46}$$

$$\begin{aligned}\dot{v}_i &= (\gamma+hB)I'v_{i+1} - (2\gamma+hB)v_i\\ &+ \gamma I'v_{i-1} + h^2 I'\lambda_i(x;\alpha)v_i\end{aligned} \tag{47}$$

$$v_{n+1} = I\vartheta_n \tag{48}$$

Noting that Eq. (47) is unstable because of the existence of the eigenvalues for all λ_i, for all i. Therefore, we follow the same previous steps in which obtained φ_i to obtain ϑ_i, by applying fuzzy FDM for TVIE we obtain:

$$\omega_0 = 0 \tag{49}$$

$$\omega_i = I'v_i - \vartheta_{i-1}(v_1, v_2, \dots, v_{i-1}) \tag{50}$$

$$\omega_{n+1} = 0 \tag{51}$$

One can find the target system as:

$$\begin{aligned}\dot{\omega}_i &= (\gamma+h\beta)I'\omega_{i+1} - (2\gamma+h\beta)\omega_i\\ &+ \gamma I'\omega_{i-1} - h^2C(x;\alpha)\omega_i\end{aligned} \tag{52}$$

We will find ϑ_i with the same steps we found φ_i , as show below:

$$\vartheta_i = \frac{I'}{(\gamma+h\beta)} \times \begin{pmatrix} (2\gamma+h\beta+h^2C)\,\vartheta_{i-1} - \gamma\vartheta_{i-2} - \\ h^2\left(CI'+\lambda_i\right)v_i + \sum_{j=1}^{i-1}\frac{\partial\vartheta_{i-1}}{\partial v_j}\dot{v}_j(v_1,v_2,\dots,v_{i-1})\end{pmatrix}$$

for i=1,2,. . . , n with Eqs. (46) to (48) we have:

$$\vartheta_1 = \frac{-I'h^2}{(\gamma+h\beta)}\left(CI'+\lambda_1\right)v_1$$

Writing the ϑ_i in the linear form

$$\vartheta_i = \sum_{j=1}^{i}\ell_{i,j}v_j$$

through simple calculations, we can derive a general recursive relationship:

$$\ell_{i,1} = \frac{-I'h^2}{\gamma+h\beta}\left(CI'+\lambda_1\right)\ell_{i-1,1} + \frac{-I'h^2}{\gamma+h\beta}\left(\ell_{i-1,2}-\ell_{i-2,1}\right)$$

$$\ell_{i,j} = \frac{-I'h^2}{\gamma+h\beta}\left(CI'+\lambda_1\right)\ell_{i-1,j} + \ell_{i-1,j-1} + \frac{-Ih^2}{\gamma+h\beta}\left(\ell_{i-1,j+1}-\ell_{i-2,j}\right)$$

where $j = 1, 2, \dots, i-2$

$$\ell_{i,i-1} = \frac{-I'h^2}{\gamma+h\beta}\left(CI'I+\lambda_{i-1}\right)\ell_{i-1,i-1} + \ell_{i-1,i-2}$$

$$\ell_{i,i} = \ell_{i-1,i-1} - \frac{+I'h^2}{\gamma+h\beta}\left(CI'+\lambda_i\right)$$

for i=1,2,3,4 with initial conditions:

$$\ell_{1,1} = \frac{-I'h^2}{\gamma+h\beta}\left(CI'+\lambda_1\right)$$

$$\ell_{2,1} = \frac{-I'h^4}{\gamma+h\beta}\left(-\frac{(2\alpha+h\beta)}{(\alpha+h\beta)}C\left(CI'I+\lambda_1\right)(I-I') - \frac{1}{\alpha+h\beta}I'\left(C\left(CI'+\lambda_1\right)+\left(CI'+\lambda_1\right)I'\lambda_1\right)\right.$$

$$\ell_{2,2} = -\frac{h^2}{\gamma+h\beta}\left(CI'+\lambda_1\right) - \frac{Ih^2}{\gamma+h\beta}\left(CI'+\lambda_2\right)$$

$$\vdots$$

$$\ell_{3,3} = \frac{-I'h^2}{\gamma+h\beta}\left(CI'+\lambda_1\right)^2 - \frac{h^2}{\gamma+h\beta}\left(C'+\lambda_2\right) - \frac{ih^2}{\gamma+h\beta}\left(C\tilde{I}+\lambda_3\right)$$

and,

$$\ell_{4,3} = \frac{-I'h^2}{(\gamma+h\beta)}\left(CI'+\lambda_3\right)\ell_{3,3} + \ell_{3,2}$$

$$\ell_{4,4} = \frac{-I'h^2}{(\gamma+h\beta)}\left(CI'+\lambda_4\right) + \ell_{3,3}$$

Let's take $\lambda_i(x) = \lambda_i$ is a TFN defined in Definitions 2 and 3, and letting $\lambda_i = [\lambda, \bar{\lambda}]$, $\quad C = [\underline{C}, \bar{C}]$ and we can solve above equation

$$\begin{aligned}\ell_{i,i-j} &= -I'\binom{i}{j+1}N_n^{j+1}\\ &- (i-j)I'\sum_{l=1}^{\left[\frac{j}{l}\right]}\frac{1}{I}\binom{j-I}{I-1}\binom{i-l}{j-2l}N_n^{j-2l+1}E_n^l\end{aligned} \tag{53}$$

For $i = 1, 2, \dots, n, j = i-1, i-2, \dots, 1, 0,$ where

$$N_n = \frac{h^2}{\gamma+h\beta}\begin{bmatrix}\frac{\lambda}{C} & \frac{C}{\lambda}\end{bmatrix}$$

$$E_n = \frac{\gamma}{\gamma+h\beta}$$

We notice that Eq. (53) is similar to Eq. (42) , then we can prove stability of Eq. (53) by Theorem 1 and Lemma (1-3).

3. Result and discussion

In this section one can taking many examples about application the results of Fuzzy RED and prove stability of

Table 1. Results solution of N_n and E_n, where $\gamma = \beta = 1$, and $h = 0.5$ For case (i)

α	$N_n = \frac{h^2}{\gamma+h\beta}\begin{bmatrix} 1+\alpha & 2-\alpha \\ 1+\alpha & 4-\alpha \end{bmatrix}$	$E_n = \frac{\gamma}{\gamma+h\beta}$
0	$N_n = \frac{0.25}{1.5}\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$	$E_n = \frac{1}{1.5}$
$\frac{1}{3}$	$N_n = \frac{0.25}{1.5}\begin{bmatrix} 1.33 & 1.66 \\ -0.66 & 3.66 \end{bmatrix}$	$E_n = \frac{1}{1.5}$
$\frac{2}{3}$	$N_n = \frac{0.25}{1.5}\begin{bmatrix} 1.66 & 1.33 \\ -0.33 & 3.33 \end{bmatrix}$	$E_n = \frac{1}{1.5}$
1	$N_n = \frac{0.25}{1.5}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$	$E_n = \frac{1}{1.5}$

Table 2. Results solution of N_n and E_n, where $\gamma = \beta = 1$, and $h = 0.5$ For case (ii)

α	$N_n = \frac{h^2}{\gamma+h\beta}\begin{bmatrix} 1+\alpha & -1+\alpha \\ 2-\alpha & 4-\alpha \end{bmatrix}$	$E_n = \frac{\gamma}{\gamma+h\beta}$
0	$N_n = \frac{0.25}{1.5}\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$	$E_n = \frac{1}{1.5}$
$\frac{1}{3}$	$N_n = \frac{0.25}{1.5}\begin{bmatrix} 1.33 & -0.66 \\ 1.66 & 3.66 \end{bmatrix}$	$E_n = \frac{1}{1.5}$
$\frac{2}{3}$	$N_n = \frac{0.25}{1.5}\begin{bmatrix} 0.833 & -0.33 \\ 1.33 & 3.33 \end{bmatrix}$	$E_n = \frac{1}{1.5}$
1	$N_n = \frac{0.25}{1.5}\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$	$E_n = \frac{1}{1.5}$

solution. Now we take two types for different fuzzy RADEs as: First, let's take at the first type of the one-dimensional fuzzy RADE as.

$$\tilde{u}_t = \gamma\tilde{u}_{xx} + \beta\tilde{u}_x + [2,3]\tilde{u}(x,t) \tag{54}$$

with initial and boundary conditions given by Eqs. (2)-(4), with target system given by

$\widetilde{W}_t = \gamma\widetilde{W}_{xx} + \beta\widetilde{W}_x - [0,1]\widetilde{W}(x,t)$,

with initial and boundary conditions presented in Eqs. (9) and (10).

Then by apply the coordinate transformation backstepping method based on of fuzzy FDM of case I, we get:

$$k_{i,i-j} = -I'\binom{i}{j+1}N_n^{j+1}$$

$$-(i-j)I\sum_{l=1}^{\left[\frac{j}{2}\right]}\frac{1}{l}\binom{j-l}{l-1}\binom{i-l}{j-2l}N_n^{j-2l+1}E_n^l$$

For $i = 1,2,\ldots,n, j = i-1, i-2,\ldots,1,0$, where $N_n = \frac{h^2}{\gamma+h\beta}\begin{bmatrix} 1+\alpha & 2-\alpha \\ -1+\alpha & 4-\alpha \end{bmatrix}$

and $E_n = \frac{\gamma}{\gamma+h\beta}$.

The results of N_n and $E_n, \forall n = 1,2,3,\ldots,\infty$, and for different values of $\alpha = 0, \frac{1}{3}, \frac{2}{3}, 1$ guarantee that the theoretical results given in Theorem 1 and 2 as well as Lemmas 1-3 are satisfied and hence the fuzzy solution is exponentially stable.

Note: When substitute result of N_n and E_n in $k_{i,i-j}$, we are getting a negative kernel. In the backstepping approach, obtaining a negative kernel is crucial for ensuring the stability of the controlled system. This concept is especially important when dealing with boundary control problems in partial differential equations (PDEs) where the goal is to stabilize the system by appropriately influencing its boundary behavior.

For case ii we have:

$$\ell_{i,i-j} = -I'\binom{i}{j+1}N_n^{j+1}$$

$$-(i-j)I'\sum_{l=1}^{\left[\frac{j}{2}\right]}\frac{1}{l}\binom{j-l}{l-1}\binom{i-l}{j-2l}N_n^{j-2l+1}E_n^l$$

for $i = 1,2,\ldots,n, j = i-1, i-2,\ldots,1,0$, where

$$N_n = \frac{h^2}{\gamma+h\beta}\begin{bmatrix} 1+\alpha & -1+\alpha \\ 2-\alpha & 4-\alpha \end{bmatrix}$$

$$E_n = \frac{\gamma}{\gamma+h\beta}$$

Similarly, as in above, for this case the results of N_n and E_n, ensures the exponentially stability of the fuzzy solution.

Now, let's take Eq. (54) of the one-dimensional fuzzy RADE as.

$\tilde{u}_t = \gamma\tilde{u}_{xx} + \beta\tilde{u}_x + [2,3]\tilde{u}(x,t)$

with initial and boundary condition Eqs. (2) to (4). And target system is,

$\widetilde{W}_t = \gamma\widetilde{W}_{xx} + \beta\widetilde{W}_x - C\widetilde{W}(x,t)$,

where C crisp constant, with initial and boundary condition Eqs. (9) and (10).

Then by apply coordinate transformation of backstepping by FDM for case I we get:

$$k_{i,i-j} = -I'\binom{i}{j+1}N_n^{j+1}$$

$$-(i-j)I'\sum_{l=1}^{\left[\frac{j}{2}\right]}\frac{1}{l}\binom{j-l}{l-1}\binom{i-l}{j-2l}N_n^{j-2l+1}E_n^l$$

for $i = 1,2,\ldots,n, j = i-1, i-2,\ldots,1,0$, where $N_n =$

Table 3. Results solution of N_n and E_n, where $\gamma = \beta = 1$, and $h = 0.5$ and $C = n, n \in \mathbb{R}$. For case (i)

α	$N_n = \frac{h^2}{\gamma+h\beta}\begin{bmatrix} 1+\alpha & n \\ n & 4-\alpha \end{bmatrix}$	$E_n = \frac{\gamma}{\gamma+h\beta}$
0	$N_n = \frac{0.25}{1.5}\begin{bmatrix} 1 & n \\ n & 4 \end{bmatrix}$	$E_n = \frac{1}{1.5}$
$\frac{1}{3}$	$N_n = \frac{0.25}{1.5}\begin{bmatrix} 1.33 & n \\ n & 3.66 \end{bmatrix}$	$E_n = \frac{1}{1.5}$
$\frac{2}{3}$	$N_n = \frac{0.25}{1.5}\begin{bmatrix} 1.66 & n \\ n & 3.33 \end{bmatrix}$	$E_n = \frac{1}{1.5}$
1	$N_n = \frac{0.25}{1.5}\begin{bmatrix} 2 & n \\ n & 3 \end{bmatrix}$	$E_n = \frac{1}{1.5}$

Table 4. Results solution of N_n and E_n, where $\gamma = \beta = 1$, and $h = 0.5$ and $C = n, n \in \mathbb{R}$. For case (ii)

α	$N_n = \frac{h^2}{\gamma+hh}\begin{bmatrix} 1+\alpha & n \\ n & 4-\alpha \end{bmatrix}$	$E_n = \frac{\gamma}{\gamma+h\beta}$
0	$N_n = \frac{0.25}{1.5}\begin{bmatrix} 1 & n \\ n & 4 \end{bmatrix}$	$E_n = \frac{1}{1.5}$
$\frac{1}{3}$	$N_n = \frac{0.25}{1.5}\begin{bmatrix} \frac{4}{3} & n \\ n & \frac{11}{3} \end{bmatrix}$	$E_n = \frac{1}{1.5}$
$\frac{2}{3}$	$N_n = \frac{0.25}{1.5}\begin{bmatrix} \frac{5}{3} & n \\ n & \frac{10}{3} \end{bmatrix}$	$E_n = \frac{1}{1.5}$
1	$N_n = \frac{0.25}{1.5}\begin{bmatrix} 2 & n \\ n & 3 \end{bmatrix}$	$E_n = \frac{1}{1.5}$

$\frac{h^2}{\gamma+h\beta}\begin{bmatrix} (1+\alpha) & C \\ C & (4-\alpha) \end{bmatrix}$ and $E_n = \frac{\gamma}{\gamma+h\beta}$

From above note the results of N_n and $E_n, \forall n = 1,2,3,$ and for different values of $\alpha = 0, \frac{1}{3}, \frac{2}{3}, 1$ guarantee that the theoretical results given in Theorem 1 and 2 as well as Lemmas $1-3$ are satisfied and hance the fuzzy solution is exponentially stable.

For case ii, one can get:

$$\ell_{i,i-j} = -I'\binom{i}{j+1}N_n^{j+1}$$

$$-(i-j)I'\sum_{l=1}^{\left[\frac{j}{2}\right]}\frac{1}{I}\binom{j-I}{I-1}\binom{i-l}{j-2l}N_n^{j-2l+1}E_n^l$$

for which $i = 1,2,\ldots,n, j = i-1, i-2, \ldots, 1, 0$ where

$$N_n = \frac{h^2}{\gamma+h\beta}\begin{bmatrix} 1+\alpha & C \\ C & 4-\alpha \end{bmatrix},$$

$$E_n = \frac{\gamma}{\gamma+h\beta}.$$

In this example one can notice that there exist now different between case i and case ii.

Similarly, as in above, for this case the results of N_n and E_n, ensures the exponentially stability of the fuzzy solution.

4. Conclusions

In this work, a new approach has been introduced for designing fuzzy boundary feedback controller fraction for the unstable fuzzy RADE based on fuzzy coordinate backstepping transformation through applying fuzzy FDM. Also, it is notable that the lower and upper function appears in the same equation. Therefore, we transform Eqs. (15) to (20), and Eqs. (44) and (45) for case i and case ii respectively into matrix differential equation. The resulting fuzzy kernel and control plant will dependent on the α-level. The obtained results are exponential stable due to the application of Theorems 1 and 2 as well as Lemmas 1-3.

For future study, one can take γ and β to be FNs and can consider different method of fuzzy backstepping approach.

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References

[1] S. S. Chang and L. A. Zadeh, (1972) *"On fuzzy mapping and control"* **IEEE transactions on systems, man, and cybernetics** (1): 30–34.

[2] C. Travis and G. Webb, (1974) *"Existence and stability for partial functional differential equations"* **Transactions of the American Mathematical Society** ***200***: 395–418.

[3] A. Balogh and M. Krstic, (2004) *"Stability of partial difference equations governing control gains in infinite-dimensional backstepping"* **Systems & control letters** ***51***(2): 151–164. DOI: 10.1016/S0167-6911(03)00222-6.

[4] M. A. Alqudah, R. Ashraf, S. Rashid, J. Singh, Z. Hammouch, and T. Abdeljawad, (2021) *"Novel numerical investigations of fuzzy Cauchy reaction–diffusion models via generalized fuzzy fractional derivative opera-*

tors" **Fractal and Fractional** *5*(4): 151. DOI: 10.3390/fractalfract5040151.

[5] A. Smyshlyaev and M. Krstic. "Lyapunov adaptive boundary control for parabolic PDEs with spatially varying coefficients". In: *2006 American Control Conference*. IEEE. 2006, 41–48. DOI: 10.1109/ACC.2006.1655328.

[6] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter. *Representation and control of infinite dimensional systems*. ***1***. Springer, 2007. DOI: 10.1007/978-0-8176-4581-6.

[7] R. F. Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory*. ***21***. Springer Science & Business Media, 2012.

[8] W. Liu, (2003) *"Boundary feedback stabilization of an unstable heat equation"* **SIAM journal on control and optimization** *42*(3): 1033–1043. DOI: 10.1137/S0363012902402414.

[9] M. Krstic and A. Smyshlyaev. *Boundary control of PDEs: A course on backstepping designs*. SIAM, 2008. DOI: 10.5555/1378241.

[10] D. Tsubakino, M. Krstic, and S. Hara. "Backstepping control for parabolic pdes with in-domain actuation". In: *2012 American Control Conference (ACC)*. IEEE. 2012, 2226–2231. DOI: 10.1109/ACC.2012.6315358.

[11] J. M. Belman-Flores, D. A. Rodríguez-Valderrama, S. Ledesma, J. J. García-Pabón, D. Hernández, and D. M. Pardo-Cely, (2022) *"A review on applications of fuzzy logic control for refrigeration systems"* **Applied Sciences** *12*(3): 1302. DOI: 10.3390/app12031302.

[12] P. Chotikunnan, R. Chotikunnan, A. Nirapai, A. Wongkamhang, P. Imura, and M. Sangworasil, (2023) *"Optimizing Membership Function Tuning for Fuzzy Control of Robotic Manipulators Using PID-Driven Data Techniques"* **Journal of Robotics and Control (JRC)** *4*(2): 128–140. DOI: 10.18196/jrc.v4i2.18108.

[13] D. N. Nguyen and T. A. Nguyen, (2024) *"Fuzzy backstepping control to enhance electric power steering system performance"* **IEEE Access** *12*: 88681–88695. DOI: 10.1109/ACCESS.2024.3419001.

[14] D. Yu, S. Ma, Y.-J. Liu, Z. Wang, and C. P. Chen, (2023) *"Finite-time adaptive fuzzy backstepping control for quadrotor UAV with stochastic disturbance"* **IEEE Transactions on Automation Science and Engineering** *21*(2): 1335–1345. DOI: 10.1109/TASE.2023.3282661.

[15] S. Salahshour, (2011) *"Nth-order fuzzy differential equations under generalized differentiability"* **Journal of fuzzy set Valued Analysis** ***2011***: 1–14. DOI: 10.5899/2011/jfsva-00043.

[16] L. Shao, J. Sun, and X. Hui. *Fuzzy Systems, Knowledge Discovery and Natural Computation Symposium: FSKDNC 2013*. DEStech Publications, Inc, 2013.

[17] G. J. McRae, W. R. Goodin, and J. H. Seinfeld, (1982) *"Numerical solution of the atmospheric diffusion equation for chemically reacting flows"* **Journal of Computational Physics** *45*(1): 1–42. DOI: 10.1016/0021-9991(82)90101-2.

[18] A. J. Perumpanani, J. A. Sherratt, and P. K. Maini, (1995) *"Phase differences in reaction-diffusion-advection systems and applications to morphogenesis"* **IMA Journal of Applied Mathematics** *55*(1): 19–33. DOI: 10.1093/IMAMAT/55.1.19.

[19] N. Emken and C. Engwer, (2020) *"A Reaction–Diffusion–Advection Model for the Establishment and Maintenance of Transport-Mediated Polarity and Symmetry Breaking"* **Frontiers in Applied Mathematics and Statistics** *6*: 570036. DOI: 10.3389/fams.2020.570036.

[20] O. M. Atyia, F. S. Fadhel, and M. H. Alobaidi, (2023) *"Using Variational Iteration Method for Solving Linear Fuzzy Random Ordinary Differential Equations."* **Mathematical Modelling of Engineering Problems** *10*(4): DOI: 10.18280/mmep.100442.

[21] F. S. Fadhel, J. H. Eidi, H. M. Wali, et al., (2021) *"Contraction mapping theorem in partial fuzzy metric spaces"* **Journal of Applied Science and Engineering** *25*(2): 353–360. DOI: 10.6180/jase.202204_25(2).0020.

[22] S. Abbasbandy and T. Hajjari, (2009) *"A new approach for ranking of trapezoidal fuzzy numbers"* **Computers & mathematics with applications** *57*(3): 413–419. DOI: 10.1016/j.camwa.2008.10.090.

[23] H. Razzaq and F. S. Fadhel, (2023) *"Variational Iteration Approach for Solving Two-Points Fuzzy Boundary Value Problems"* **Al-Nahrain Journal of Science** *26*(3): 51–59. DOI: 10.22401/anjs.26.3.08.

[24] H.-J. Zimmermann, (2010) *"Fuzzy set theory"* **Wiley interdisciplinary reviews: computational statistics** *2*(3): 317–332. DOI: 10.1002/wics.82.

[25] L. Stefanini, (2010) *"A generalization of Hukuhara difference and division for interval and fuzzy arithmetic"* **Fuzzy sets and systems** *161*(11): 1564–1584. DOI: 10.1016/j.fss.2009.06.009.

[26] T. Allahviranloo and S. Gholami, (2012) *"Note on Generalized Hukuhara differentiability of interval-valued functions and interval differential equations"* **Journal of fuzzy set Valued Analysis *2012***(4): DOI: 10.5899/2012/JFSVA-00135.

[27] T. Allahviranloo, Z. Gouyandeh, A. Armand, and A. Hasanoglu, (2015) *"On fuzzy solutions for heat equation based on generalized Hukuhara differentiability"* **Fuzzy sets and systems *265***: 1–23. DOI: 10.1016/j.fss.2014.11.009.

[28] I. K. HANAN, M. Z. AHMAD, and F. S. FADHEL, (2017) *"The Backstepping Method For Stabilizing Time Fractional Order Partial Differential Equation"* **Journal of Theoretical & Applied Information Technology *95***(6):

[29] A. Balogh and M. Krstic. "Infinite-step backstepping for a heat equation-like PDE with arbitrarily many unstable eigenvalues". In: *Proceedings of the 2001 American Control Conference.(Cat. No. 01CH37148)*. ***3***. IEEE. 2001, 2480–2485. DOI: 10.1109/ACC.2001.946125.

[30] D. M. Bošković, A. Balogh, and M. Krstić, (2003) *"Backstepping in infinite dimension for a class of parabolic distributed parameter systems"* **Mathematics of Control, Signals and Systems *16***: 44–75. DOI: 10.1007/s00498-003-0128-6.

[31] I. K. Hanan, M. Z. Ahmad, F. S. Fadhel, and G. R. Mohammed, (2018) *"Backstepping in infinite dimensional for the time fractional order partial differential equations"* **Malaysian Journal of Fundamental and Applied Sciences *14***(1): 83–89. DOI: 10.11113/MJFAS.V14N1.930.