Enlightenment Of Heat Diffusion Using New Homotopy Perturbation Method

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In this research paper, we obtained the analytic exact solution of time dependent nonhomogeneous one-dimensional heat conduction equation by using new homotopy perturbation method. The obtained solution of heat diffusion equation was explained graphically using MATLAB. The numerical values of the solution of Heat equation are shown in a table. The novelty of the procedure is that it does no longer require small parameter in an equation and obtain the analytical solution without converting the problem into homogeneous boundary condition. We concluded that the solution of nonlinear and linear differential equation can be received through using new homotopy perturbation method. Conclusion of this study have super utility in the discipline of engineering, mathematics, biomedical and many others.

Keywords: Nonlinear equation; Heat diffusion; New homotopy perturbation method; boundary and initial conditions

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1. Introduction

Nowadays, problems in the field of Biomedical, Biology, Physics, Engineering as well as industries are modelled by linear and nonlinear partial differential equations (PDE). The exact or approximate solution of PDE can be obtained by several numerical and analytical methods. Perturbation is an analytical approximation method, which depends on small and large parameters essentially exist in the equation. But this method has certain limitation. To reduce the limitation of Perturbation method, in 1992, Shi-jun Liao introduced homotopy analysis method (HAM) for both small and large parameters [1]. We know that several nonlinear problems do not have small parameters. To determine the small parameters is superior art for nonlinear problems. Ji-Huan He derived Homotopy Perturbation Method (HPM), to investigate the linear and non-linear ordinary and partial differential equations, by unification of perturbation and homotopy techniques [2–8]. HPM do not require small parameters in the equation and resolve the limitations of perturbation techniques.

Recently Biazar and Eslami [9], proposed the new homotopy perturbation method (NHPM). There are two Important steps of NHPM, first step is formation of suitable homotopy equation and second step is decision of relevant initial approximation reckon. The study divulged that with less computational work NHPM is the most powerful technique, which can be used to obtain analytical solution of various kinds of linear and nonlinear PDE's. This method is widely used by researchers to obtain solution of various functional equations [10–20]. In NHPM, we constructed a suitable homotopy by decomposition of source function with less determinate and made calculations in simpler form unlike other perturbation methods. The acquired results immediately mean the fact that NHPM is very protuberant as contrast to HPM or some other perturbation technique. To establish exact solution of linear and nonlinear problem with boundary and initial condition, NHPM is most suitable method to apply. In the present research paper, NHPM is used to obtain the analytic solution for one dimensional heat conduction equation with boundary and initial conditions which is identical as the universally accepted exact solution.

2. Theory and formula

Perturbation method depends on small parameter. Selection of unsuitable small parameter can lead to wrong solutions [21]. On the other hand, Homotopy is an important part of topology [22] and it can convert any non-linear problem in to a finite linear problem and it doesn't depend on small parameter [2, 4, 14]. To introduce the NHPM in the above-mentioned references, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

with the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{2}$$

where A is a general differential operator, B is a boundary operator, f(r) is a known analytical function, and Γ is the boundary of domain Ω. The operator A in (1) can be rewritten as a sum of L and N as follows

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega, \tag{3}$$

where L and N are linear and nonlinear part of A.

By the homotopy technique, we construct the Homotopy perturbation $\mathrm{H}(v, p) : \Omega \times (0, 1) \rightarrow R$

$$\mathrm{H}(v, p) = L(v) - L(u_0) + pL(u_0) + p(N(v) - f(r)) = 0 \tag{4}$$

where p is an embedding parameter, u_0 is an initial approximation of (1). The embedding parameter p and initial approximation u_0 satisfies the boundary conditions. As p changes from zero to unity, $v(r, p)$ changes from u_0 to u(r).

3. Experimental setup

The heat diffusion is the most important phenomena, to obtain the exact solution of one-dimensional heat conduction equation

$$u_t - u_{xx} = x \tag{5}$$

with boundary conditions $u(0, t) = 2, 0 \leq x \leq 1, t > 0, u(1, t) = t, t > 0$ and initial condition $u(x, 0) = 3\sin 2\pi x + 2(1 - x)$ is deciphered in this research paper, by applying NHPM $(1 - p)\left(\frac{\partial v}{\partial t} - u_0\right) + p\left(\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - x\right) = 0$.

Integrating both the sides with respect to t

$$v(x, t) = \left[\int_0^t u_0 dt + v(x, 0)\right] - p\int_0^t \left(u_0 - \frac{\partial^2 v}{\partial x^2} - x\right) dt \tag{6}$$

where $u_0(x, t) = \sum_{n=0}^{\infty} c_n(x) p_n(t), v(x, t) = \sum v_i p^i$ and $p_i(t) = t^i$ in the equation (6).

From given boundary conditions, we have $v(x, 0) = u(x, 0) = 3\sin 2\pi x + 2(1 - x)$ then in equation (6) the coefficients of p are $p^0 : v_0 = \int_0^t (c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots + c_n t^n)\, dt + 3\sin 2\pi x + 2(1 - x), p^1 : v_1 = -\int_0^t u_0 dt + \int_0^t \left(\frac{\partial^2 v_0}{\partial x^2}\right) dt + \int_0^t x dt, \quad p^2 : v_2 = \int_0^t \left(\frac{\partial^2 v_1}{\partial x^2}\right) dt, p^3 : v_3 = \int_0^t \left(\frac{\partial^2 v_2}{\partial x^2}\right) dt$ and soon $p^n : v_n = \int_0^t \left(\frac{\partial^2 v_{n-1}}{\partial x^2}\right) dt, \quad p^{n+1} : \int_0^t \left(\frac{\partial^2 v_n}{\partial x^2}\right) dt = 0.$

Now we will obtain the value of $v_1(x, t)$ such that the values of $v_2, v_3, \cdots v_n$ will be vanish.

$$-\int_0^t u_0 dt + \int_0^t \left(\frac{\partial^2 v_0}{\partial x^2}\right) dt + \int_0^t x dt = 0 \tag{7}$$

Now putting the coefficients of t equal to zero in equation (7), which gives the values of $c_0 = x - 12\pi^2 \sin 2\pi x, c_1 = c_0'' = 48\pi^4 \sin 2\pi x, c_2 = \frac{c_1''}{2} = -96\pi^6 \sin 2\pi x, c_3 = \frac{c_2''}{3} = 128\pi^8 \sin 2\pi x$ and soon $c_n = \frac{c''_{(n-1)}}{(n)} = \frac{(-1)^{(n+1)} 3(2\pi)^{(2n+2)} \sin 2\pi x}{n!}$.

Moreover, from equation (6)

$$\begin{aligned} v(x,t) &= 2 + (t-2)x \\ &+ 3\sin 2\pi x \left[1 - \frac{t}{1!}(2\pi)^2 + \left(\frac{t^2}{2!}\right)(2\pi)^4 \right. \\ &- \left(\frac{t^3}{3!}\right)(2\pi)^6 + \left(\frac{t^4}{4!}\right)(2\pi)^8 + \cdots \\ &\left. \ldots + (-1)^{(n+1)}\left(\frac{t^{(n+1)}}{(n+1)!}\right)(2\pi)^{(2n+2)}\right], \\ v(x,t) &= 2 + (t-2)x + 3\left(e^{-4\pi^2 t}\right)\sin 2\pi x \end{aligned} \tag{8}$$

4. Result discussions

From the above solution procedure, we obtained exact solution of one-dimensional heat conduction equation with boundary and initial conditions which is similar to the solution found in Herman [23]. It was also received with the aid of other mathematical approaches viz. Dirichlet boundary conditions, Fourier series, Boundary value Green function and so forth., but the traditional methods are more complicated to drive the solution instead of NHPM. In traditional methods, firstly nonhomogeneous boundary conditions problem needed to be converted an auxiliary problem with homogeneous boundary conditions. Here, we obtained analytic solution by NHPM while other methods including HPM gives the semi-analytic or approximate solutions for the problem. Furthermore, from equation (8) we computed the numerical values $v(x, t)$ for $0 \leq x \leq 1 \& 0 \leq t \leq 1$ which is shown in the table 01.

Table 1. Comparison of time performance and performance with different superpixel methods

x\t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
0.1	3.5634	1.8440	1.8207	1.8300	1.8400	1.8500	1.8600	1.8700	1.8800	1.8900	1.9000
0.2	4.4532	1.6751	1.6411	1.6600	1.6800	1.7000	1.7200	1.7400	1.7600	1.7800	1.8000
0.3	4.2532	1.4851	1.4611	1.4900	1.5200	1.5500	1.5800	1.6100	1.6400	1.6700	1.7000
0.4	2.9634	1.2740	1.2807	1.3200	1.3600	1.4000	1.4400	1.4800	1.5200	1.5600	1.6000
0.5	1.0000	1.0500	1.1000	1.1500	1.2000	1.2500	1.3000	1.3500	1.4000	1.4500	1.5000
0.6	-0.9634	0.8260	0.9193	0.9800	1.0400	1.1000	1.1600	1.2200	1.2800	1.3400	1.4000
0.7	-2.2532	0.6149	0.7389	0.8100	0.8800	0.9500	1.0200	1.0900	1.1600	1.2300	1.3000
0.8	-2.4532	0.4249	0.5589	0.6400	0.7200	0.8000	0.8800	0.9600	1.0400	1.1200	1.2000
0.9	-1.5634	0.2560	0.3793	0.4700	0.5600	0.6500	0.7400	0.8300	0.9200	1.0100	1.1000
1	0.0000	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	0.7000	0.8000	0.9000	1.0000

Moreover, we plotted a graph Figure 1 for the values of Table 1 with the help of MATLAB

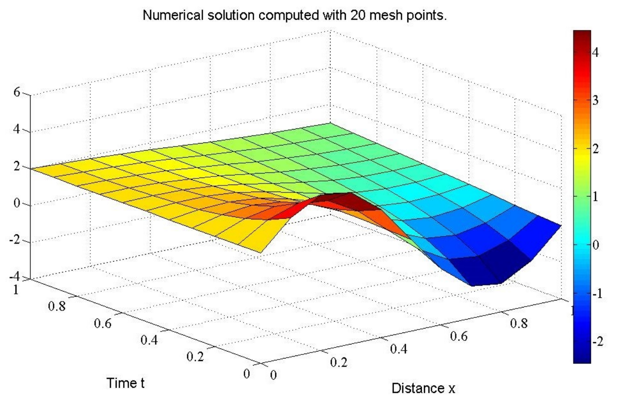

Fig. 1

A standard graph Figure 1 was also obtained by using MATLAB, which is in accordance as the graph obtained in other mathematical methods. This graph illustrates the Heat diffusion in x direction with respect to time. The Heat diffusion is dependent on time and distance. It is observed that increasing the distance the diffusivity treated as sine function, and it also satisfies the boundary conditions $u(0,t) = 2, 0 \leq x \leq 1, t > 0, u(1,t) = t, t > 0$. The values in Table 01 also shown the contemplate incredibility of NHPM for linear and non-linear partial differential equation with boundary and initial conditions. From physical points of view, several linear and nonlinear PDEs do not have small parameters, so NHPM is most suitable to apply and can be used to obtain analytical solution.

5. Conclusions

In this paper, we obtained the exact solution of time dependent nonhomogeneous one-dimensional heat conduction equation, with boundary and initial conditions, by using NHPM without converting the problem into homogeneous boundary condition. We enumerated the exact solution very rapidly as compared to homotopy perturbation method (HPM) and other traditional methods. We observed that NHPM is very distinguished, when accuracy, capability and reliability have a key role to play. NHPM is less computational technique in comparison to others and most powerful technique.

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