A Generalization Of Exponentiated Pareto-I Distribution With Applications

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Received: June 24 2023; Accepted: September 09 2023

According to earlier research, transmuting a standard distribution often results in a compound distribution that performs better and is more flexible. In light of this fact, this article proposes two novel generalized versions of the exponentiated Pareto-I distribution, called cubic transmuted exponentiated Pareto-I and fourth rank transmuted exponentiated Pareto-I by using the generalization formula for transmuted distribution. Some statistical properties are derived. Model parameters are estimated using the maximum likelihood method. Finally, an application of the two proposed distributions to two real data sets with diverse shapes is illustrated and compared with some distributions based on the exponential family and the exponentiated Pareto-I distribution. The applications suggest that the fourth version performs better than the cubic one for all shapes of the distribution. The new exponentiated Pareto-I models exhibit constant, upside-down, and bathtub hazard rates. The justification for the practicality of the new lifetime models is based on their ability to model real-life data sets from different perspectives.

Keywords: Pareto Distribution; Cubic Transmutation; Fourth Rank Transmutation; Maximum likelihood Estimation; Moment Generating Function

http://dx.doi.org/10.6180/jase.202405_27(5).0004

1. Introduction

Vilfredo Pareto (1848–1923) came up with the Pareto distribution, which is used to model the distribution of income and other financial variables and to describe social and other events. In recent years, people have tried to explain many real-world things using the Pareto distribution or a form that is like it. For more details, see Johnson et al. [1].

Gupta et al. [2] introduced the exponentiated Pareto distribution with two parameters, showed that this distribution can be used quite effectively in analyzing many lifetime data sets. Nadarajah [3] proposed five different forms of exponential Pareto, respectively, called EP-1, EP-II, EP-III, generalized EP, and truncated EP, and derived some statistical properties for them. Shawky and Abu-Zinadah [4] looked at different ways to estimate different parameters of the exponentiated Pareto distribution. Shaw and Buckley [5] introduced a quadratic rank transmutation map that has been used by Fatima and Roohi [6] to propose a transmuted exponentiated Pareto-I distribution. Eledum and Ansari [7] developed a new generalization of exponentiated Pareto called the MG Exponentiated Pareto distribution based on MG transmutation map [8]. Aryal and Tsokos [9] introduced another version of the quadratic rank transmutation map and utilized it to build a flexible family of probability distributions, with the extreme value distribution serving as the base value distribution.

Granzottoa et al. [10] made the cubic ranking transmutation map, also called the transmuted distribution of second order, with two shape parameters. Many researchers have used this method to develop novel distributions known as cubic transmuted distributions. For examples, cubic rank transmuted Kumaraswamy [11], cubic rank transmuted inverse rayleigh [12], cubic rank transmuted generalized Gompertz [13], and cubic rank transmuted exponentiated Pareto-I [14]. Rahman et al. [15] developed a general transmuted family of distributions with emphasis on the cubic transmuted family and used it to propose a cubic transmuted Pareto distribution [16]. This method used by Eledum [14], to propose the cubic rank transmuted exponentiated Pareto-I. Al-Kadim [17] developed a general form for the transmuted map that proposed by Shaw and Buckley [5], and this form was used by Ansari and Eledum [18] to introduce another version for the cubic transmuted Pareto distribution.

Frequently, transforming a standard distribution yields a compound distribution with superior performance and greater adaptability. Therefore, the purpose of this study is to explore the potential benefits of transforming standard distributions, building upon previous research that highlighted improved performance and flexibility in compound distributions post-transmutation. Specifically, the research introduced two new variations of the exponentiated Pareto-I distribution using the generalization formula for transmuted distributions.

In this paper, we use a generalization formula for the transmuted distribution developed by Al-Kadim [17] to propose two new generalized versions of the exponentiated Pareto-I distribution. These two versions of Pareto-1 distribution are called the cubic transmuted exponentiated Pareto–I (CTEP-I) distribution and the fourth transmuted exponentiated Pareto–I (FTEP-I) distribution respectively. Some statistical properties are derived. Model parameters are estimated by the maximum likelihood method. Finally, an application of CTEP-I and FTEP-I to two real data sets with different shapes is illustrated and compared with some distributions based on the exponential family as well as the exponentiated Pareto-I distribution.

We are motivated by the new exponentiated Pareto–I models because they exhibit a constant, upside-down, and bathtub hazard rates, as illustrated in Figures 5 and 6. The justification for the practicality of the new lifetime models is based on their ability to model real-life data sets from different perspectives, as illustrated in Section 4. We used the new exponentiated Pareto–I models since they have a wide ability for modeling different shapes of real data sets; this claim has been demonstrated in the applications section.

1.1. The Exponentiated Pareto-I Distribution

Let X be a random variable with the exponentiated Pareto-I distribution. The probability density function (pdf) and the cumulative distribution function (cdf) are defined, respectively, as

$$g(x) = \alpha b^{\alpha} e^{-\alpha x}; \quad x \in [\ln b, \infty); \quad \alpha, b > 0 \tag{1}$$

and

$$G(x) = 1 - b^{\alpha} e^{-\alpha x}; \quad x \in [\ln b, \infty); \quad \alpha, b > 0 \tag{2}$$

where α, b are the shape and location parameters, respectively.

1.2. Cubic Ranking Transmutation Map

According to the generalization formula for transmuted distribution developed by Al-Kadim [17], the cdf and pdf of the cubic transmuted distribution are given in Eqs. (3) and (4), respectively.

$$F(x) = (1+\lambda)G(x) - 2\lambda G^2(x) + \lambda G^3(x), |\lambda| \leq 1 \tag{3}$$

$$f(x) = (1+\lambda)g(x) - 4\lambda g(x)G(x) + 3\lambda G^2(x)g(x) \tag{4}$$

where $G(x)$ and $g(x)$ refer to the cdf and pdf of the base distribution, respectively. Observe that at $\lambda = 0, F(x) = G(x)$ and $f(x) = g(x)$.

The remainder of this paper is organized as follows: The first new proposed distribution, cubic transmuted exponentiated Pareto-I (CTEP-I), is presented in Section 2. Section 3 pertains to the second new proposed distribution of forth rank transmuted exponentiated Pareto–I (FTEP–I). An application of the CTEP-I and FTEP-I to two real data sets for illustration is conducted in Section 4. Finally, Section 5 gives some concluding remarks.

2. Cubic transmuted exponentiated pareto-i distribution

This section pertains to the new proposed distribution cubic transmuted exponentiated Pareto-1 (CTEP-I) distribution. Discusses the probability density function (pdf), cumulative distribution function (cdf), hazard function, and some statistical properties, and parameters estimates.

2.1. Density and Cumulative Function for CTEP-I Distribution

Lemma 1. Let X be a random variable with CTEP-I distribution. The cdf and pdf are defined, respectively, as

$$F(x) = 1 - b^{\alpha} e^{-\alpha x} \left(1 - \lambda b^{\alpha} e^{-\alpha x} + \lambda b^{2\alpha} e^{-2\alpha x}\right) \tag{5}$$

and

$$\begin{aligned} f(x) &= \alpha b^{\alpha} e^{-\alpha x} - 2\lambda \alpha b^{2\alpha} e^{-2\alpha x} + 3\lambda \alpha b^{3\alpha} e^{-3\alpha x} \\ &= \alpha b^{\alpha} e^{-\alpha x}\left(1 - 2\lambda b^{\alpha} e^{-\alpha x} + 3\lambda b^{2\alpha} e^{-2\alpha x}\right) \\ & x \in [\ln b, \infty); \quad \alpha, b > 0, \quad |\lambda| \leq 1 \end{aligned} \tag{6}$$

where α, b and λ are the shape, location, and transmuted parameters respectively.

Special case: for $b = 1$ we get cubic transmuted exponential distribution. You can get Eqs. (5) and (6) by putting Eq. (2) into Eq. (3) and Eq. (1) into Eq. (4), respectively.

Figures 1, 2, 3, and 4 illustrate some of the possible shapes for the pdf of the CTEP-I distribution for selected values of the parameters α, b and λ.

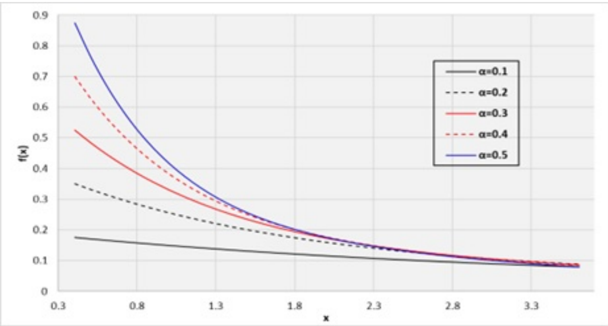

Fig. 1. The $f(x)$ of CTEP-I at $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, \lambda = 0.75$ and $b = 1.5$.

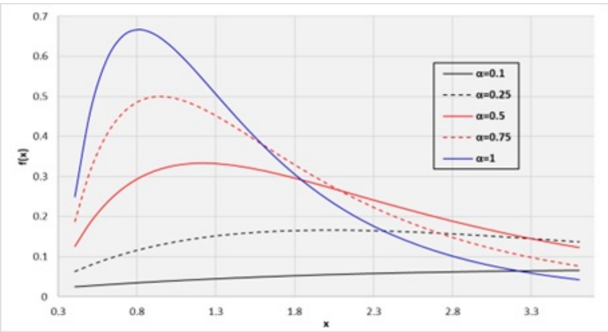

Fig. 2. The $f(x)$ of CTEP-I at $\alpha = 0.1, 0.25, 0.5, 0.75, 1, \lambda = -0.75$ and $b = 1.5$.

From the plot of the pdf of Figures reffig:1, 2 and 3, we can observe that, as the shape parameter α increases, the skewness of the distribution fixes, and as the transmuted parameter λ increases the skewness of the distribution increases, this also can be seen in Table 1. Moreover, Figure 4 shows that the parameter b is the location one. It is also observed that the distribution is moderate to highly right-skewed depending on the transmuted parameter λ.

Lemma 2. The limit of CTEP-I distribution density as $x \to \ln b$ is $\alpha(1+\lambda)$ and the limit as $x \to \infty$ is 0 .

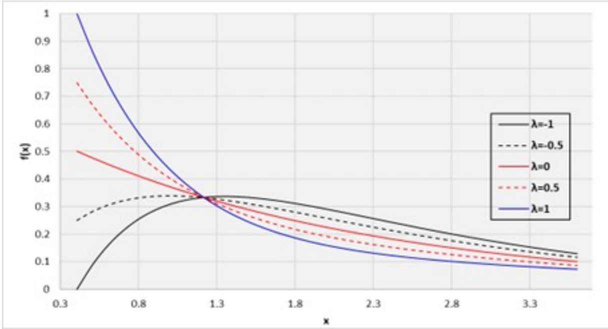

Fig. 3. The $f(x)$ of CTEP-I at $\lambda = -1, -0.5, 0, 0.5, 1, \alpha = 0.5$ and $b = 1.5$

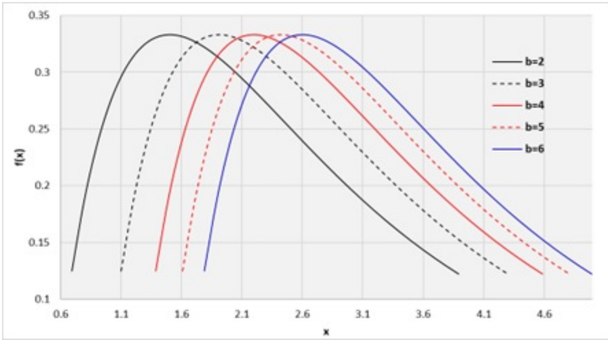

Fig. 4. The $f(x)$ of CTEP-I at $\lambda = -0.75, \alpha = 0.5$ and $b = 2, 3, 4, 5, 6$

Proof. The proof is straightforward.

Lemma 3. $f(x)$ of Eq. (6) is a probability density function.

To prove $f(x)$ is a pdf, we need to prove $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

Proof of $f(x) \geq 0$

Substitute $x = \ln b$ in Eq. (6) we get

$$\begin{aligned} f(x) &= \alpha b^{\alpha} e^{-\alpha \ln b}\left(1 - 2\lambda b^{\alpha} e^{-\alpha \ln b} + 3\lambda b^{2\alpha} e^{-2\alpha \ln b}\right) \\ &= \alpha b^{\alpha} e^{\ln b^{-\alpha}}\left(1 - 2\lambda b^{\alpha} e^{\ln b^{-\alpha}} + 3\lambda b^{2\alpha} e^{\ln b^{-2\alpha}}\right) \\ &= \alpha b^{\alpha} b^{-\alpha}\left(1 - 2\lambda b^{\alpha} b^{-\alpha} + 3\lambda b^{2\alpha} b^{-2\alpha}\right) = \alpha(1+\lambda) \end{aligned}$$

Since $\alpha > 0, |\lambda| \leq 1$ then $\alpha(1+\lambda) \geq 0$.

Similarly, substitute $x = \infty$ in Eq. (6) we get, $f(x) = 0$

Proof of $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{\ln b}^{\infty} f(x;\alpha,b,\lambda)dx$$
$$= \int_{\ln b}^{\infty}\left(\alpha b^{\alpha}e^{-\alpha x} - 2\lambda\alpha b^{2\alpha}e^{-2\alpha x} + 3\lambda\alpha b^{3\alpha}e^{-3\alpha x}\right)dx$$
$$= \alpha b^{\alpha}\int_{\ln b}^{\infty} e^{-\alpha x}dx - 2\lambda\alpha b^{2\alpha}\int_{\ln b}^{\infty} e^{-2\alpha x}dx + 3\lambda\alpha b^{3\alpha}\int_{\ln b}^{\infty} e^{-3\alpha x}dx$$
$$= \alpha b^{\alpha}\left[\left(\frac{e^{-\alpha x}}{-\alpha}\right)\right]_{\ln b}^{\infty} - 2\lambda\alpha b^{2\alpha}\left[\left(\frac{e^{-2\alpha x}}{-2\alpha}\right)\right]_{\ln b}^{\infty} + 3\lambda\alpha b^{3\alpha}\left[\left(\frac{e^{-3\alpha x}}{-3\alpha}\right)\right]\ln b$$
$$= -b^{\alpha}\left[\left(e^{-\alpha x}\right)\right]_{\ln b}^{\infty} + \lambda b^{2\alpha}\left[\left(e^{-2\alpha x}\right)\right]_{\ln b}^{\infty} - \lambda b^{3\alpha}\left[\left(e^{-3\alpha x}\right)\right]_{\ln b}^{\infty}$$
$$= 1 + \lambda - \lambda = 1$$

This proves that $f(x)$ is a probability density function.

2.2. Survival and Hazard Functions

The survival (reliability) function of CTEP-1 distribution is

$$S(x) = 1 - F(x)$$
$$S(x) = b^{\alpha}e^{-\alpha x}\left(1 - \lambda b^{\alpha}e^{-\alpha x} + \lambda b^{2\alpha}e^{-2\alpha x}\right) \tag{7}$$

The hazard function of CTEP-1 distribution is

$$h(x) = \frac{f(x)}{S(x)}$$
$$h(x) = \frac{\alpha\left(1-2\lambda b^{\alpha}e^{-\alpha x}+3\lambda b^{2\alpha}e^{-2\alpha x}\right)}{1-\lambda b^{\alpha}e^{-\alpha x}+\lambda b^{2\alpha}e^{-2\alpha x}} \tag{8}$$

Figures 5 and 6 show the hazard function of CTEP-1 distribution for different values of α and λ at $b = 1.5$.

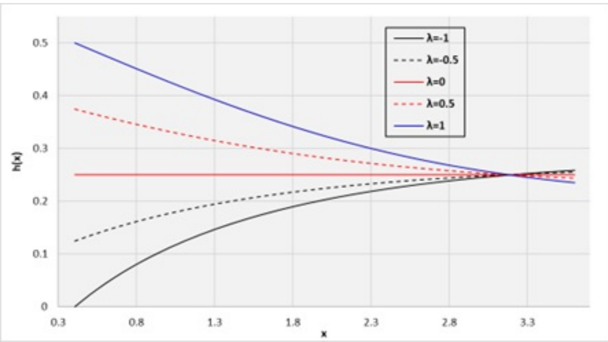

Fig. 5. The $h(x)$ of CTEP-I at $\lambda = -1, -0.5, 0.0, 0.5, 1, \alpha = 0.25$ and $b = 1.5$.

From Figures 5 and 6, we can see that for negative values of λ, the hazard rate is increasing, and for positive values of λ, it is decreasing; further, it moves up as the parameter α increases. Moreover, the hazard function for the CTEP-1 distribution is a constant when the transmuted parameter λ is zero.

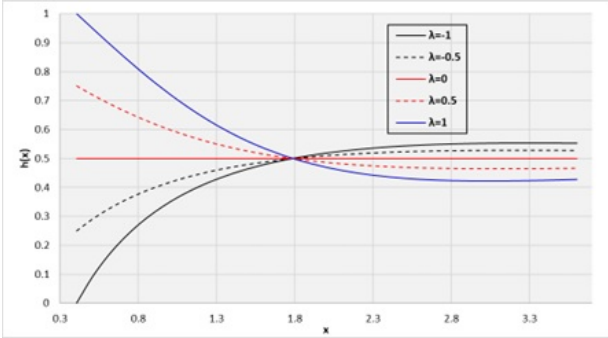

Fig. 6. The $h(x)$ of CTEP-I at $\lambda = -1, -0.5, 0.0, 0.5, 1, \alpha = 0.5$ and $b = 1.5$.

2.3. Statistical Properties

In this subsection, some statistical properties for the proposed distribution, CTEP-1 distribution are demonstrated. These properties include moments, moment generating function, median, mode, geometric mean, and harmonic mean.

2.3.1. *The Moments*

Theorem 1. If X is a random variable having the CTEP-1 distribution, then the r^{th} moment of X about the origin is

$$E\left(X^r\right) = b^{\alpha}\alpha^{-r}\Gamma(r+1,\alpha\ln b) - \lambda b^{2\alpha}(2\alpha)^{-r}\Gamma(r+1,2\alpha\ln b) + \lambda b^{3\alpha}(3\alpha)^{-r}\Gamma(r+1,3\alpha\ln b) \tag{9}$$

Proof. We know that $E\left(X^r\right) = \int_{-\infty}^{\infty} x^r f(x)dx$

Substitute $f(x)$ in the above equation by its value in Eq. (6) to get

$$E\left(X^r\right) = \int_{\ln b}^{\infty} x^r\left[\alpha b^{\alpha}e^{-\alpha x}\left(1 - 2\lambda b^{\alpha}e^{-\alpha x} + 3\lambda b^{2\alpha}e^{-2\alpha x}\right)\right]dx$$
$$= \alpha b^{\alpha}\int_{\ln b}^{\infty} x^r\left(e^{-\alpha x} - 2\lambda b^{\alpha}e^{-2\alpha x} + 3\lambda b^{2\alpha}e^{-3\alpha x}\right)dx$$
$$= \alpha b^{\alpha}\left(\int_{\ln b}^{\infty} x^r e^{-\alpha x}dx - 2\lambda b^{\alpha}\int_{\ln b}^{\infty} x^r e^{-2\alpha x}dx + 3\lambda b^{2\alpha}\int_{\ln b}^{\infty} x^r e^{-3\alpha x}dx\right)$$
$$= \alpha b^{\alpha}\left(I_1 - 2\lambda b^{\alpha}I_2 + 3\lambda b^{2\alpha}I_3\right)$$

where

$I_1 = \int_{\ln b}^{\infty} x^r e^{-\alpha x}dx$, $I_2 = \int_{\ln b}^{\infty} x^r e^{-2\alpha x}dx$ and $I_3 = \int_{\ln b}^{\infty} x^r e^{-3\alpha x}dx$ now $I_1 = \int_{\ln b}^{\infty} x^r e^{-\alpha x}dx$

let $\alpha x = u$ then $x = \frac{u}{\alpha}$ and $dx = \frac{1}{\alpha}du$ for $x = \ln b$ then $u = \alpha\ln b$ and for $x = \infty$ then $u = \alpha\ln\infty = \infty$

$I_1 = \frac{1}{\alpha^{r+1}}\int_{\alpha\ln b}^{\infty} u^r e^{-u}du = \frac{1}{\alpha^{r+1}}\int_{\alpha\ln b}^{\infty} u^{(r+1)-1}e^{-u}du = \frac{1}{\alpha^{r+1}}\Gamma(r+1,\alpha\ln b)$

hence $I_1 = \frac{1}{\alpha^{r+1}}\Gamma(r+1,\alpha\ln b)$, where $\Gamma(r+1,\alpha\ln b)$ is an incomplete gamma function. following the same way, we get

$I_2 = \frac{1}{(2\alpha)^{r+1}}\Gamma(r+1, 2\alpha\ln b)$ and $I_3 = \frac{1}{(3\alpha)^{r+1}}\Gamma(r+1, 3\alpha\ln b)$ by performing some algebra calculations, we get

$$E(X^r) = b^\alpha \alpha^{-r}\Gamma(r+1, \alpha\ln b) - \lambda b^{2\alpha}(2\alpha)^{-r}\Gamma(r+1, 2\alpha\ln b) + \lambda b^{3\alpha}(3\alpha)^{-r}\Gamma(r+1, 3\alpha\ln b)$$

and this completes the proof.

2.3.2. The Moment Generating Function

Theorem 2. If X is a random variable having the CTEP-1 distribution, then the moment generating function of X is

$$M_X(t) = \frac{\alpha b^t}{\alpha - t} - \frac{2\lambda\alpha b^t}{2\alpha - t} + \frac{3\lambda\alpha b^t}{3\alpha - t} \tag{10}$$

Furthermore, the mean and variance of the distribution are given as $E(X) = \ln b + \frac{1}{\alpha} - \frac{\lambda}{6\alpha}$ and $\mathrm{Var}(X) = -\frac{36+2\lambda-\lambda^2}{36\alpha^2}$

Proof. We know that $M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$

Substitute $f(x)$ in the above equation by its value in Eq. (6) to get $M_X(t)$

$$= \int_{\ln b}^{\infty} e^{tx}\left[\alpha b^\alpha e^{-\alpha x}\left(1 - 2\lambda b^\alpha e^{-\alpha x} + 3\lambda b^{2\alpha} e^{-2\alpha x}\right)\right]dx$$

$$= \alpha b^\alpha \int_{\ln b}^{\infty} e^{-(\alpha - t)x}dx$$

$$- 2\lambda\alpha b^{2\alpha}\int_{\ln b}^{\infty} e^{-(2\alpha - t)x}dx + 3\lambda\alpha b^{3\alpha}\int_{\ln b}^{\infty} e^{-(3\alpha - t)x}dx$$

$$= \alpha b^\alpha I_1 - 2\lambda\alpha b^{2\alpha} I_2 + 3\lambda\alpha b^{3\alpha} I_3$$

where $I_1 = \int_{\ln b}^{\infty} e^{-(\alpha-t)x}dx$, $I_2 = \int_{\ln b}^{\infty} e^{-(2\alpha-t)x}dx$ and $I_3 = \int_{\ln b}^{\infty} e^{-(3\alpha-t)x}dx$ now

$$I_1 = \int_{\ln b}^{\infty} e^{-(\alpha-t)x}dx = -\left[\frac{e^{-(\alpha-t)x}}{\alpha - t}\right]_{\ln b}^{\infty}$$

$$= -\left(\frac{e^{-\infty} - e^{-(\alpha-t)\ln b}}{\alpha - t}\right) \quad \text{following the}$$

$$= \frac{e^{\ln b^{-(\alpha-t)}}}{\alpha - t} = \frac{b^{-\alpha}b^t}{\alpha - t}; \quad (\alpha - t) > 0$$

same way, we get $I_2 = \frac{b^{-2\alpha}b^t}{2\alpha - t}, (2\alpha - t) > 0$ and $I_3 = \frac{b^{-3\alpha}b^t}{3\alpha - t}, \quad (3\alpha - t) > 0$

therefore $M_X(t) = \frac{\alpha b^t}{\alpha - t} - \frac{2\lambda\alpha b^t}{2\alpha - t} + \frac{3\lambda\alpha b^t}{3\alpha - t}$ The mean and variance can be easily obtained by differentiating Eq. (10) i times (i = 1, 2) with respect to t and setting $t = 0$.

2.3.3. The Mode

Theorem 3. Mode of CTEP-1 distribution is given as

$$x = -\frac{1}{\alpha}\ln\left(\frac{2\lambda - \sqrt{\lambda(4\lambda - 9)}}{9\lambda b^\alpha}\right); \quad \lambda < 0 \tag{11}$$

Proof. The mode or modal value of a continuous random variable X with a probability density function $f(x)$ is the value of x for which $f(x)$ takes a maximum value, that is, $f'(x) = 0$

Differentiate $f(x)$ in Eq. (6) to get

$$f'(x) = -\alpha^2 b^\alpha e^{-\alpha x} + 4\lambda\alpha^2 b^{2\alpha} e^{-2\alpha x} - 9\lambda\alpha^2 b^{3\alpha} e^{-3\alpha x}$$

$$= -\alpha^2 b^\alpha\left(e^{-\alpha x} - 4\lambda b^\alpha e^{-2\alpha x} + 9\lambda b^{2\alpha} e^{-3\alpha x}\right)$$

Equate

$$e^{-\alpha x} - 4\lambda b^\alpha e^{-2\alpha x} + 9\lambda b^{2\alpha} e^{-3\alpha x} = 0$$
$$9\lambda b^{2\alpha} e^{-3\alpha x} - 4\lambda b^\alpha e^{-2\alpha x} + e^{-\alpha x} = 0$$
$$e^{-\alpha x}\left(9\lambda b^{2\alpha} e^{-2\alpha x} - 4\lambda b^\alpha e^{-\alpha x} + 1\right) = 0$$
$$9\lambda b^{2\alpha} e^{-2\alpha x} - 4\lambda b^\alpha e^{-\alpha x} + 1 = 0$$

Let $b^\alpha e^{-\alpha x} = y$ then $9\lambda y^2 - 4\lambda y + 1 = 0$ we know that $Ay^2 + By + C = 0 \quad y = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ then

$$y = \frac{4\lambda \pm \sqrt{16\lambda^2 - 36\lambda}}{18\lambda}$$

$$b^\alpha e^{-\alpha x} = \frac{2\lambda \pm \sqrt{\lambda(4\lambda - 9)}}{9\lambda}$$

$$e^{-\alpha x} = \frac{2\lambda \pm \sqrt{\lambda(4\lambda - 9)}}{9\lambda b^\alpha}$$

$$x = -\frac{1}{\alpha}\ln\left(\frac{2\lambda \pm \sqrt{\lambda(4\lambda - 9)}}{9\lambda b^\alpha}\right)$$

For the positive value of x we have

$$x = -\frac{1}{\alpha}\ln\left(\frac{2\lambda - \sqrt{\lambda(4\lambda - 9)}}{9\lambda b^\alpha}\right)$$

provided that the ln :.exists.

2.3.4. Geometric Mean

Theorem 4. Geometric mean of CTEP-1 distribution is given as

$$G = \mathrm{Antilog}\left\{\ln(\ln b) + b^\alpha\Gamma(0, \alpha\ln b) - \lambda b^{2\alpha}\left[\Gamma(0, 2\alpha\ln b) + b^\alpha\Gamma(0, 3\alpha\ln b)\right]\right\} \tag{12}$$

where Antilog is the inverse method for calculating the logarithm of the same number, and $\Gamma(0, \alpha\ln b)$, $\Gamma(0, 2\alpha\ln b)$ and $\Gamma(0, \alpha 3\ln b)$ are an incomplete gamma functions.

Proof. The logarithm of the geometric mean of a continuous random variable X with a probability density function $f(x)$ is, $\log G = \int_{-\infty}^{\infty} \log x f(x)dx$

Substitute the pdf of CTEP-1 distribution of Eq. (6) in the above equation to get

$$\log G = \int_{\ln b}^{\infty} \log x\left(\alpha b^\alpha e^{-\alpha x} - 2\lambda\alpha b^{2\alpha} e^{-2\alpha x} + 3\lambda\alpha b^{3\alpha} e^{-3\alpha x}\right)dx$$

$$= \alpha b^\alpha \int_{\ln b}^{\infty} e^{-\alpha x}\log x dx - 2\lambda\alpha b^{2\alpha}\int_{\ln b}^{\infty} e^{-2\alpha x}\log x dx$$

$$+ 3\lambda\alpha b^{3\alpha}\int_{\ln b}^{\infty} e^{-3\alpha x}\log x dx$$

$$= \alpha b^\alpha I_1 - 2\lambda\alpha b^{2\alpha} I_2 + 3\lambda\alpha b^{3\alpha} I_3$$

where $I_1 = \int_{\ln b}^{\infty} e^{-\alpha x}\log x dx$, $\quad I_2 = \int_{\ln b}^{\infty} e^{-2\alpha x}\log x dx$ and $I_3 = \int_{\ln b}^{\infty} e^{-3\alpha x}\log x dx$ now

$$I_1 = \int_{\ln b}^{\infty} e^{-\alpha x}\log x dx = \left[(\log x)\frac{e^{-\alpha x}}{-\alpha}\right]_{\ln b}^{\infty}$$

$$- \int_{\ln b}^{\infty}\frac{1}{x}\frac{e^{-\alpha x}}{-\alpha}dx = \frac{\ln(\ln b)}{\alpha b^\alpha} + \frac{1}{\alpha}\Gamma(0, \alpha\ln b)$$

following the same way, we get

$I_2 = \frac{\ln(\ln b)}{2\alpha b^{2\alpha}} + \frac{1}{2\alpha}\Gamma(0, 2\alpha\ln b)$ and $I_3 = \frac{\ln(\ln b)}{3\alpha b^\alpha} + \frac{1}{3\alpha}\Gamma(0, 3\alpha\ln b)$

hence

$$\log G = \ln(\ln b) + b^{\alpha}\Gamma(0, \alpha \ln b) - \lambda b^{2\alpha}\left[\Gamma(0, 2\alpha \ln b) + b^{\alpha}\Gamma(0, 3\alpha \ln b)\right]$$

finally

$$G = \text{ Antilog } \left\{\ln(\ln b) + b^{\alpha}\Gamma(0, \alpha \ln b) - \lambda b^{2\alpha}\left[\Gamma(0, 2\alpha \ln b) + b^{\alpha}\Gamma(0, 3\alpha \ln b)\right]\right\}$$

and this completes the proof.

2.3.5. *Harmonic Mean*

Theorem 5. The harmonic mean of CTEP-1 distribution is given as

$$H = \Big[\alpha b^{\alpha}\Gamma(0, \alpha \ln b) - 2\lambda\alpha b^{2\alpha}\Gamma(0, 2\alpha \ln b) + 3\lambda\alpha b^{3\alpha}\Gamma(0, 3\alpha \ln b)\Big]^{-1} \tag{13}$$

Proof. The proof is straightforward.
Hint: Use the formula $H^{-1} = \int_{-\infty}^{\infty} \frac{1}{x} f(x)dx$

The mean, median, mode, skewness, and variance of the CTEP-1 distribution for various values of α, λ at $b = 1.5$ are given in Table 1.

From Table 1, regarding the CTEP-I distribution, we can observe that, for positive values of the transmuted parameter λ, the mode does not exist (see Figure 1), while, for negative values of λ, the median is not defined. On the other hand, the measures of central tendency decrease with the transmuted parameter λ. While the coefficient of skewness and variance increase. Holding other CTEP-1's parameters fixed, the measures of central tendency and the variance go down with α, but the coefficient of skewness stays the same.

2.3.6. *Parameters Estimation*

Maximum likelihood approach can be used for the estimation of model parameters. The maximum likelihood estimate (MLE) of the parameters that are inherent within the CTEP-I distribution are given by the following.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from CTEP-I distribution. Then the likelihood function is given by

$$\begin{aligned} L &= \prod_{i=1}^{n} f(x_i; \alpha, b, \lambda) \\ &= \prod_{i=1}^{n}\left[\alpha b^{\alpha} e^{-\alpha x_i}\left(1 - 2\lambda b^{\alpha} e^{-\alpha x_i} + 3\lambda b^{2\alpha} e^{-2\alpha x_i}\right)\right] \\ &= \alpha^n b^{n\alpha} e^{-\alpha \sum_{i=1}^{n} x_i} \prod_{i=1}^{n}\left(1 - 2\lambda b^{\alpha} e^{-\alpha x_i} + 3\lambda b^{2\alpha} e^{-2\alpha x_i}\right) \end{aligned}$$

so, the log likelihood function is

$$\ln L = n \ln \alpha + n\alpha \ln b - \alpha \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln\left(1 - 2\lambda b^{\alpha} e^{-\alpha x_i} + 3\lambda b^{2\alpha} e^{-2\alpha x_i}\right) \tag{14}$$

differentiating the log-likelihood of Eq. (14) partially with respect to the parameters α, b and λ, and then equating to zero, we get three normal Eqs (15), (16) and (17).

$$\frac{\partial \ln L}{\partial \alpha} = n + \alpha n \ln b - \alpha \sum_{i=1}^{n} x_i + 2\lambda\alpha b^{\alpha} \sum_{i=1}^{n} \frac{e^{-\alpha x_i}(\ln b - x_i)\left(3\lambda e^{-\alpha x_i} b^{\alpha} - 1\right)}{1 - 2\lambda b^{\alpha} e^{-\alpha x_i} + 3\lambda b^{2\alpha} e^{-2\alpha x_i}} = 0 \tag{15}$$

$$\frac{\partial \ln L}{\partial b} = n - 2\lambda b^{\alpha} \sum_{i=1}^{n} \frac{e^{-\alpha x_i}\left(1 + 3b^{\alpha} e^{-\alpha x_i}\right)}{1 - 2\lambda b^{\alpha} e^{-\alpha x_i} + 3\lambda b^{2\alpha} e^{-2\alpha x_i}} = 0 \tag{16}$$

$$\frac{\partial \ln L}{\partial \lambda} = \sum_{i=1}^{n} \frac{e^{-\alpha x_i}\left(2 - 3b^{\alpha} e^{-\alpha x_i}\right)}{1 - 2\lambda b^{\alpha} e^{-\alpha x_i} + 3\lambda b^{2\alpha} e^{-2\alpha x_i}} = 0 \tag{17}$$

The maximum likelihood estimates $\hat{\theta} = (\hat{\alpha}, \hat{b}, \hat{\lambda})$ of $\theta = (\alpha, b, \lambda)$ is obtained by solving the nonlinear system of Eqs (15), (16) and (17). It is more convenient to use nonlinear optimization algorithms such as the quazi-Newton or Newton-Raphson to numerically maximize the loglikelihood function in Eq. (14).

3. Fourth rank transmuted exponentiated pareto-i (ftep-i) distribution

In this section, the probability density function (pdf) and cumulative distribution function (cdf) for the new proposed distribution fourth rank transmuted exponentiated Pareto-1 (FTEP-I) distribution is explained.

3.1. Fourth Rank Transmutation Map

Based on the general formula of the transmuted distribution developed by Al-Kadim [17], a random variable X is said to have a fourth rank transmuted distribution if its cdf is given by

$$F(x) = (1 + \lambda)G(x) - \lambda G^2(x) + \lambda G^3(x) - \lambda G^4(x), \quad |\lambda| \leq 1 \tag{18}$$

and the pdf is given by

$$f(x) = g(x)\left(1 + \lambda - 2\lambda G(x) + 3\lambda G^2(x) - 4\lambda G^3(x)\right) \tag{19}$$

where $\mathrm{G}(x)$ and $\mathrm{g}(x)$ are the cdf and pdf of the base distribution respectively. Observe that at $\lambda = 0, F(x) = \mathrm{G}(x)$ and $f(x) = \mathrm{g}(x)$.

3.2. Density and Cumulative Function for FTEP-I Distribution

Lemma 4. Let X be a random variable with FTEP-I distribution. The cdf and pdf are defined, respectively, as

$$F(x) = 1 - b^{\alpha} e^{-\alpha x}\left(1 - 2\lambda + 4\lambda b^{\alpha} e^{-\alpha x} - 3\lambda b^{2\alpha} e^{-2\alpha x} + \lambda b^{3\alpha} e^{-3\alpha x}\right) \tag{20}$$

Table 1. Some measures of central tendency, skewness, and the variance of the CTEP-1 for combination values of the distribution's parameters.

	Measure	$\lambda=-1$	$\lambda=-0.5$	$\lambda=0$	$\lambda=0.5$	$\lambda=1$
	Mean	2.7388	2.5721	2.4055	2.2388	2.0721
	Median	-	-	1.7918	1.5303	1.2738
$a=0.5$	Mode	1.3524	0.9986	-	-	-
	skewness	1.9729	1.9563	2.0000	2.0928	2.2305
	Variance	3.6667	3.8611	4.0000	4.0833	4.1111
	Mean	1.5721	1.4888	1.4055	1.3221	1.2388
$a=1$	Median	-	-	1.0986	0.9679	0.8396
	Mode	0.8789	0.7020	-	-	-
	skewness	1.9729	1.9563	2.0000	2.0928	2.2305
	Variance	0.9167	0.9653	1.0000	1.0208	1.0278
	Mean	0.9888	0.9471	0.9055	0.8638	0.8221
	Median	-	-	0.7520	0.6867	0.6226
$a=2$	Mode	0.6422	0.5537	-	-	-
	skewness	1.9729	1.9563	2.0000	2.0928	2.2305
	Variance	0.2292	0.2413	0.2500	0.2552	0.2569
	Mean	0.7944	0.7666	0.7388	0.7110	0.6832
$a=3$	Median	-	-	0.6365	0.5929	0.5502
	Mode	0.5633	0.5043	-	-	-
	skewness	1.9729	1.9563	2.0000	2.0928	2.2305
	Variance	0.1019	0.1073	0.1111	0.1134	0.1142

$$f(x)=\alpha b^{\alpha}e^{-\alpha x}\left(1-2\lambda+8\lambda b^{\alpha}e^{-\alpha x}\right.$$
$$\left.-9\lambda b^{2\alpha}e^{-2\alpha x}+4\lambda b^{3\alpha}e^{-3\alpha x}\right) \quad (21)$$
$$x\in[\ln b,\infty),\quad \alpha,b>0,\quad |\lambda|\le 1$$

where α, b and λ are the shape, location, and transmuted parameters respectively.

Special case: for $b=1$ we get fourth rank transmuted exponential distribution. You can get Eqs. (20) and (21) by putting Eq. (2) into Eq. (18) and Eq. (1) into Eq. (19), respectively.

Figures 7, 8, and 9 illustrate some of the possible shapes for the pdf of FTEP-I distribution for selected values of the parameters, α, b and λ.

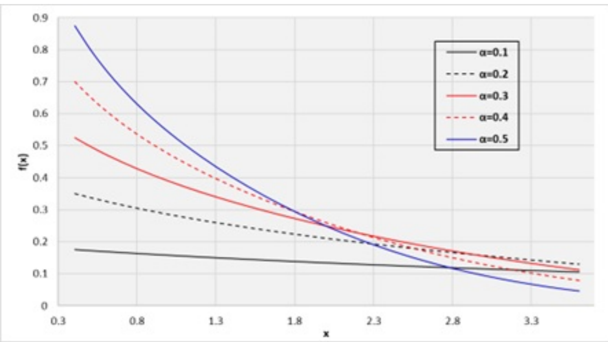

Fig. 7. The $f(x)$ of FTEP-I at $\alpha=0.1,0.2,\ldots,0.5,\lambda=0.75$ and $b=1.5$.

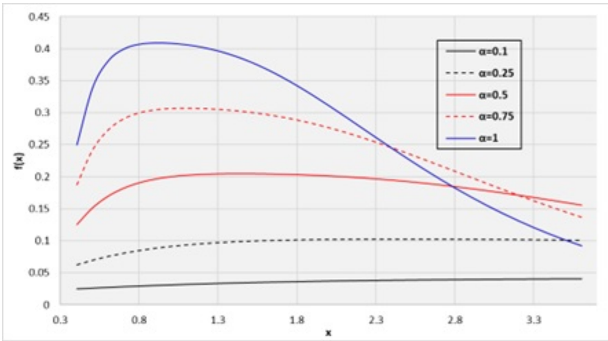

Fig. 8. The $f(x)$ of FTEP-I at $\alpha=0.1,0.25,0.5,0.75,1,\lambda=-0.75$ and $b=1.5$.

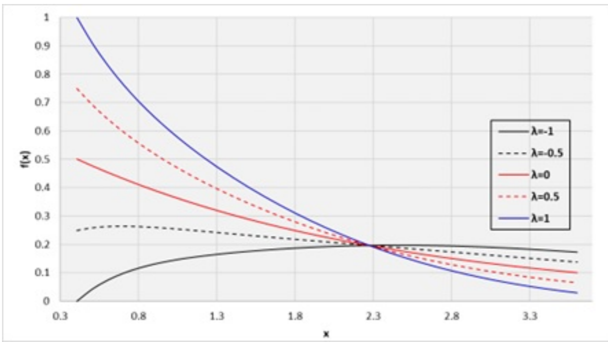

Fig. 9. The $f(x)$ of FTEP-I at $\lambda=-1,-0.5,0,0.5,1,\alpha=0.5$ and $b=1.5$

Lemma 5. The limit of FTEP-I distribution density as $x\to\ln b$ is $\alpha(1+\lambda)$ and the limit as $x\to\infty$ is 0.

Proof. The proof is straightforward.

Lemma 6. $f(x)$ of Eq. (21) is a probability density function.

To prove $f(x)$ is a pdf, we need to prove $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

Proof of $f(x) \geq 0$

Substitute $x = \ln b$ in Eq. (21) we get $f(x) = \alpha b^{\alpha} e^{-\alpha \ln b}\left(1 - 2\lambda b^{\alpha} e^{-\alpha \ln b} + 3\lambda b^{2\alpha} e^{-2\alpha \ln b}\right)$

$=\alpha b^{\alpha} e^{\ln b^{-\alpha}}\left(1 - 2\lambda b^{\alpha} e^{\ln b^{-\alpha}} + 3\lambda b^{2\alpha} e^{\ln b^{-2\alpha}}\right)$

$= \alpha b^{\alpha} b^{-\alpha}\left(1 - 2\lambda b^{\alpha} b^{-\alpha} + 3\lambda b^{2\alpha} b^{-2\alpha}\right)$

$= \alpha(1 - 2\lambda + 3\lambda) = \alpha(1+\lambda)$

Since $\alpha > 0, |\lambda| \leq 1$ then $\alpha(1+\lambda) \geq 0$.

Similarly, substitute $x = \infty$ in Eq. (21) we get $f(x) = 0$.

Proof of $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{\ln b}^{\infty} f(x;\alpha,b,\lambda)dx$$

$$= \int_{\ln b}^{\infty}\left(\alpha b^{\alpha} e^{-\alpha x} - 2\lambda\alpha b^{2\alpha} e^{-2\alpha x} + 3\lambda\alpha b^{3\alpha} e^{-3\alpha x}\right)dx$$

$$= \alpha b^{\alpha}\int_{\ln b}^{\infty} e^{-\alpha x}dx - 2\lambda\alpha b^{2\alpha}\int_{\ln b}^{\infty} e^{-2\alpha x}dx$$

$$+ 3\lambda\alpha b^{3\alpha}\int_{\ln b}^{\infty} e^{-3\alpha x}dx$$

$$= \left[\alpha b^{\alpha}\left(\frac{e^{-\alpha x}}{-\alpha}\right)\right]_{\ln b}^{\infty} - 2\lambda\alpha b^{2\alpha}\left[\left(\frac{e^{-2\alpha x}}{-2\alpha}\right)\right]_{\ln b}^{\infty}$$

$$+ 3\lambda\alpha b^{3\alpha}\left[\left(\frac{e^{-3\alpha x}}{-3\alpha}\right)\right]\ln b$$

$$= \left[-b^{\alpha}\left(e^{-\alpha x}\right)\right]_{\ln b}^{\infty} + \lambda b^{2\alpha}\left[\left(e^{-2\alpha x}\right)\right]_{\ln b}^{\infty}$$

$$- \lambda b^{3\alpha}\left[\left(e^{-3\alpha x}\right)\right]_{\ln b}^{\infty} = 1 + \lambda - \lambda$$

$$= 1$$

3.3. Parameters Estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from FTEP-I distribution. Then the log likelihood function is given by

$$\ln L = n\ln\alpha + n\alpha\ln b - a\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\ln\left(\begin{array}{c} 1 - 2\lambda + 8\lambda b^{\alpha} e^{-\alpha x_i} - 9\lambda b^{2\alpha} e^{-2\alpha x_i} \\ +4\lambda b^{3\alpha} e^{-3\alpha x_i}\end{array}\right) \quad (22)$$

Therefore, the maximum likelihood estimates of α, b and λ which maximize Eq. (22), must satisfy the three normal equations (23), (24) and (25).

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{i=1}^{n}\frac{2\lambda b^{\alpha} e^{-\alpha x_i}\left(\ln b - x_i\right)\left(4 - 9b^{\alpha} e^{-\alpha x_i} + 6b^{2\alpha} e^{-2\alpha x_i}\right)}{1 - 2\lambda + 8\lambda b^{\alpha} e^{-\alpha x_i} - 9\lambda b^{2\alpha} e^{-2\alpha x_i} + 4\lambda b^{3\alpha} e^{-3\alpha x_i}} + \frac{n}{\alpha} + n\ln b - \sum_{i=1}^{n} x_i = 0 \quad (23)$$

$$\frac{\partial \ln L}{\partial b} = \sum_{i=1}^{n}\frac{2\lambda b^{\alpha} e^{-\alpha x_i}\left(4 - 9b^{\alpha} e^{-\alpha x_i} + 6b^{2\alpha} e^{-2\alpha x_i}\right)}{1 - 2\lambda + 8\lambda b^{\alpha} e^{-\alpha x_i} - 9\lambda b^{2\alpha} e^{-2\alpha x_i} + 4\lambda b^{3\alpha} e^{-3\alpha x_i}} + \frac{an}{b} = 0 \quad (24)$$

$$\frac{\partial \ln L}{\partial \lambda} = \sum_{i=1}^{n}\frac{-2 + b^{\alpha} e^{-\alpha x_i}\left(8 - 9b^{\alpha} e^{-\alpha x_i} + 4b^{2\alpha} e^{-2\alpha x_i}\right)}{1 - 2\lambda + 8\lambda b^{\alpha} e^{-\alpha x_i} - 9\lambda b^{2\alpha} e^{-2\alpha x_i} + 4\lambda b^{3\alpha} e^{-3\alpha x_i}} = 0 \quad (25)$$

The maximum likelihood estimates $\hat{\theta} = (\hat{\alpha}, \hat{b}, \hat{\lambda})$ of $\theta = (\alpha, b, \lambda)$ is obtained by solving the nonlinear system of Eqs. (23), (24) and (25).

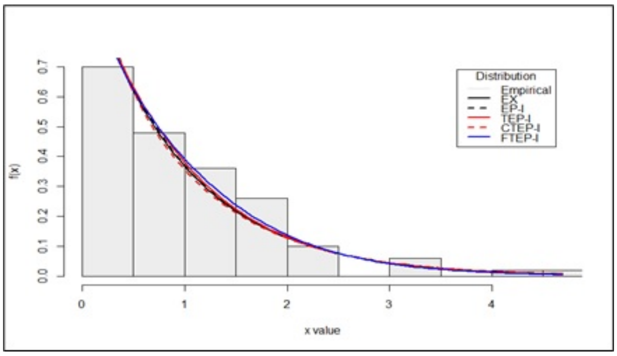

Fig. 10. Histogram of FTK data set and the fitted pdfs.

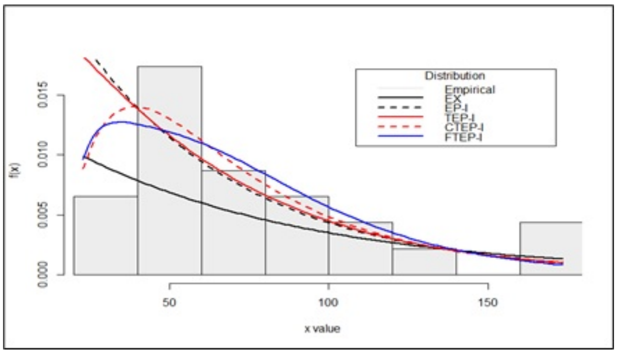

Fig. 11. Histogram of RBB data set and the fitted pdfs.

4. Applications

In this section, we provide an application of the two proposed distributions, CTEP-1 and FTEP-I. Moreover, CTEP-1 and FTEP-I are compared with some related distributions, including the exponential distribution, exponentiated Pareto-I (EP-I), and transmuted exponentiated Pareto-I (TEP-I) distribution [6]. For this purpose, we consider

Table 2. Kevlar 49/epoxy strand failure time data (pressure at 90%).

0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.04
0.05	0.06	0.07	0.07	0.08	0.09	0.09	0.10
0.10	0.11	0.11	0.12	0.13	0.18	0.19	0.20
0.23	0.24	0.24	0.29	0.34	0.35	0.36	0.38
0.40	0.42	0.43	0.52	0.54	0.56	0.60	0.60
0.63	0.65	0.67	0.68	0.72	0.72	0.72	0.73
0.79	0.79	0.80	0.80	0.83	0.85	0.90	0.92
0.95	0.99	1.00	1.01	1.02	1.03	1.05	1.10
1.10	1.11	1.15	1.18	1.20	1.29	1.31	1.33
1.34	1.40	1.43	1.45	1.50	1.51	1.52	1.53
1.54	1.54	1.55	1.58	1.60	1.63	1.64	1.80
1.80	1.81	2.02	2.05	2.14	2.17	2.33	3.03
3.03	3.34	4.20	4.69				

Table 3. ML parameter estimates, model selection criteria and k-s statistic for FTK data set.

Distribution	Parameter estimates	$-2\log(\mathrm{L})$	AIC	BIC	$\mathrm{k}-\mathrm{s}$	p-value
Exponential	$\hat{\lambda} = 1.048$	191.0424	193.024	195.6475	0.089	0.4061
EP-I	$\hat{\alpha} = 1.057$	190.939	193.939	195.544	0.0908	0.3805
TEP-I	$\hat{\alpha} = 1.0439$ $\hat{\lambda} = -0.083$	188.8291	192.829	198.039	0.0954	0.3218
CTEP-I (proposed)	$\hat{\alpha} = 1.0439$ $\hat{\lambda} = 0.127$	188.668	192.668	197.878	0.0986	0.2848
FTEP-I (proposed)	$\hat{\alpha} = 1.252$ $\hat{\lambda} = -0.237$	188.322	192.322	197.533	0.0998	0.2745

Table 4. Revolutions for each ball bearing data set.

22.88	28.92	33.00	41.52	42.12	45.6
48.8	51.84	51.96	54.12	55.56	67.8
68.44	72.64	78.88	84.12	93.12	98.64
105.12	105.84	127.92	160.04	173.4	

two real data sets. These data sets exhibit various shapes of the distribution, including leptokurtic and moderately unimodal, right-skewed data sets, demonstrating the flexibility of the two distributions for fitting real-world data. For the purpose of the analysis, we set $\ln b = \exp[\min(x)]$.

In order to compare the two proposed distributions CTEP-I and FTEP-I with other related distributions, several goodness of fit measures have been taken into consideration, including -2×log-likelihood (-2log(L)), Akaike's information criterion (AIC), Schwarz's Bayesian information criterion (BIC), and the Kolmogorov-Smirnov (k-s) test and its corresponding p-value. The computation of these statistics is determined by the following formulas:

$$AIC = -2\log(L) + 2k \tag{26}$$

$$BIC = -2\log(L) + k\ln n \tag{27}$$

$$k - s = \sup\left(|F(x) - F_n(x)|\right) \tag{28}$$

where k represents the number of the estimated parameters, n refers to the sample size, and $\log(L)$ denotes the log-likelihood of the data.

4.1. Failure Time of Kevar 49/epoxy (FTK)

The data set in Table 2 reported by Barlow et al. [19] represents the failure time of Kevlar 49/epoxy strands when the pressure is at 90% stress level (FTK). This data is leptokurtic, unimodal, and right-skewed (skewness = 1.626, kurtosis = 6.699), analyzed by Andrews and Herzberg [20], Cordeiro and Lemonte [21] and Al-Aqtash et al. [22]. Table 3 displays the results of fitting this data to the exponential distribution, EP-I distribution, TEP-I distribution, and the two suggested distributions, the CTEP-I and FTEP-I.

By comparing the goodness of fit statistics (k-s and p-value) in Table 3 among the five distributions, all distributions are competitors and fit the leptokurtic unimodal, right-skewed data very well (see Figure 10). Based on -2log(L) and AIC criteria, the two proposed distributions, CTEP-I and FTEP-I, perform better than other ones. This application indicates that the CTEP-I and FTEP-I distributions are a good match for leptokurtic, unimodal, and right-skewed data.

4.2. Number of million revolutions for each ball bearing before failing (RBB)

The data set in Table 4 is reported by Lawless [23] and has recently been used by Ieren et al. [24] and Ieren and Abdullahi [25]. This data set represents the number of million revolutions for each of the 23 ball bearings in the

Table 5. ML parameter estimates, model selection criteria and k-s statistic for RBB data set.

Distribution	Parameter estimates	$-2\log(L)$	AIC	BIC	$k-s$	p-value
Exponential	$\hat{\lambda}=0.013$	244.264	246.246	247.399	0.298	0.0265
EP-I	$\hat{\alpha}=0.019$	229.373	231.373	232.508	0.173	0.4459
TEP-I	$\hat{\alpha}=0.021$ $\hat{\lambda}=-0.120$	226.911	230.911	233.182	0.164	0.5154
CTEP-I (proposed)	$\hat{\alpha}=0.021$ $\hat{\lambda}=-0.575$	225.966	229.966	232.237	0.106	0.9316
FTEP-I (proposed)	$\hat{\alpha}=0.029$ $\hat{\lambda}=-0.669$	225.520	229.520	231.791	0.097	0.9661

life tests before failing (RBB). The RBB data set is approximately symmetric, unimodal, and right- skewed (skewness= 1.0145 and kurtosis= 3.37). Table 5 demonstrates the results.

Concerning the RBB data set, from Table 5, the values of the k-s and their corresponding p-values indicate that all distributions fit the data set adequately at a 5% significance level. On the other hand, the suggested distributions CTEP-I and FTEP-I best fit the RBB data set because they have the minimum values of all model selection criteria: -2log(L), AIC, and BIC (see Figure 11). This application suggests that both CTEP-I and FTEP-I distributions well fit the unimodal symmetric (mesokurtic) data sets.

5. Conclusions

In this paper, new two generalized versions of the exponentiated Pareto-I distribution which called the cubic transmuted exponentiated Pareto-I (CTEP-I) and the fourth rank transmuted exponentiated Pareto–I (FTEP-I) distributions respectively are introduced by using the generalization formula for transmuted distribution proposed by Al-Kadim (2018). Some statistical properties of the two distributions are derived. The model parameters are estimated by the maximum likelihood method. Finally, an application of CTEP-I and FTEP-I distributions to some real data sets and compared with some distributions based on exponential family and exponentiated Pareto-I distribution is explained. We concluded that the two proposed distributions, CTEP-1 and FTEP-I, fit the unimodal leptokurtic and right-skewed data, as well as the unimodal mesokurtic data. Generally, the FTEP-I distribution performs better than the CTEP-1 distribution. It is suggested that CTEP-1 and FTEP-I be used to model right-skewed data and data sets with constant, upside-down, and bathtub hazard rates, and it is hoped that they will receive significant applications in the future.

Acknowledgment

The authors would like to thank the editor and the anonymous reviewers for their insightful remarks and recommendations, which have significantly enhanced this article.

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