Embedding b-Metric Spaces Of Reducible Fuzzy Digraphs Into Normed Spaces

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Received: Oct. 12, 2024; Accepted: Apr. 10, 2025

In traditional and iterative algebraic techniques, representing fuzzy digraphs using their adjacency matrices can pose challenges, particularly when dealing with graphs featuring an extensive quantity of nodes or edges. These difficulties are especially pronounced in graphs that do not necessarily exhibit strong connectivity between nodes, which are known as reducible digraphs. In this study, we explore different structure of reducible fuzzy digraphs, employing an innovative b-metric structure. This structure enhances the embedding of massive data entities or nodes into low-dimensional realm depicted by a normed space. Therefore, our aim is to define a novel notion of a distance function called b_{RD}-distance between any two vertices (or nodes) in reducible fuzzy digraphs and utilize it to introduce quasi-pseudo- b-metric spaces for these digraphs. Furthermore, to minimize b_{RD}-distance calculations, we demonstrate the process of embedding such b_{RD}-metrics on reducible fuzzy graphs into the designated normed space ℓ_∞. Ultimately, a computational example, real-world application, and comparative evaluation confirm the practicality of the suggested technique in handling reducibility, asymmetry, and vagueness in fuzzy digraphs.

Keywords: b_{RD}-Distance; Quasi-pseudo- b_{RD}-metric; Embedding b_{RD}-metric into normed space; Fuzzy graph; Reducible graph.

http://dx.doi.org/10.6180/jase.202602_29(2).0016

1. Introduction

Directed graphs (or digraph) are commonly used to model relations (arcs or edges) with a specific direction between points (nodes or vertices), while undirected graphs are employed when the relations between points are symmetric and have no direction. The theory of a directed graph has become an energetic mathematical tool for modelling many problems in real world subjects including computer science [1, 2], social network analysis [3], biology [4], and chemistry [5]. Similarly, fuzzy set theory has garnered considerable attention in recent academic investigations [6–12]. Modeling these problems frequently requires multi-polar information, including uncertainty and ambiguity in predictions. Therefore, the use of fuzzy graphs to describe these problems is highly beneficial. The theory of fuzzy graphs was originally presented by Kauffman [13], but its development was furthered by Rosenfeld [14]. For more details on fuzzy graphs, refer to [15–19]. Addressing these problems frequently requires handling multi-polar information, including uncertainty and ambiguity in predictions.

A digraph D_G can be assigned by an adjacency matrix B that is a common algebraic technique to specify a digraph in some situations with a finite number of nodes or edges [20].

For a non-negative matrix $B = \left\{b_{ij}\right\}_{i,j\in\{1,2,\ldots,n\}}$, the related digraph $D_G(B)$ of B is the directed graph with node set $\{1, 2, \ldots, n\}$ and with edge set described as follows: (i, j) is a directed edge if and only if $b_{ij} > 0$ [21]. A matrix B is irreducible if and only if its related digraph $D_G(B)$ is strongly connected; a matrix B is reducible if and only if $D_G(B)$ is not irreducible, i.e., B is reducible if and only if $D_G(B)$ is not strongly connected, see [22–25]. The difference between the properties of irreducibility and reducibility for digraphs makes the investigation of the latter of special interest, and characterizing not strongly connected digraphs with a huge number of nodes or edges is a nontrivial problem. However, this algebraic narration of digraphs with large or complex structures cannot capture all the details about their characteristics. This motivates us to investigate the metric characteristics of a reducible fuzzy digraph by employing functional techniques to elucidate its structures, without relying on the adjacency matrix of the digraph, as used in traditional methods. Moreover, to the best of our knowledge, the reducibility of digraphs has received limited attention in the context of real-world problems, which may not necessarily exhibit strong connectivity and are therefore represented by reducible digraphs.

This motivates the use of b-metrics in fuzzy reducible digraphs due to their capacity to control the flexibility of the triangular inequality, model asymmetric distances, and analyze fuzzy path structures and connectivity. The relaxation of the classical b-metric constraints, combined with the incorporation of a tunable parameter denoted as M, offers a generalized distance measure capable of capturing the inherent uncertainty, vagueness, and asymmetry that characterize fuzzy digraphs. By tackling the distinctive challenges presented by reducible digraphs and their structural constraints in large-scale data objects, this study extends embedding theories into uncharted territory. It bridges a theoretical gap and offers practical insights for graph-related applications, especially in contexts where reducibility and asymmetry are essential considerations.

In the present work, the first goal is to explore a new categorization of reducible fuzzy digraphs, denoted as $RD_{\mathcal{FG}}(V,E)$, according to b-metric spaces. This exploration consolidates the theoretical realm of $RD_{\mathcal{FG}}(V,E)$ with the notion of b-metric by introducing the concept of a distance function, which will be named as b_{RD}-distance in $\mathrm{RD}_{\mathcal{FG}}(V,E)$. Consequently, a normal question arises: What types of distances can be classified by this b_{RD}-distance. In other words, what is the classification of reducible fuzzy digraphs that can be made using the b-metric structures. This leads to the establishment and identification of a modern metric for the digraph $RD_{\mathcal{FG}}(V,E)$, termed quasi-pseudo- b_{RD}-metric space of $RD_{\mathcal{FG}}(V,E)$.

In the context of a given b_{RD}-distance function or b_{RD}-metric on the digraph $RD_{\mathcal{FG}}(V,E)$, it is natural to explore the embedding of such b_{RD}-metric structures. The second goal is to minimize distance calculations, especially when the b_{RD}-distance function is computationally expensive for complex data objects. A contemporary approach to achieving this is by embedding the set of objects into points in a low-dimensional space. Therefore, our focus is on embedding methods that generate a mapping based on the b_{RD}-distance function, which is required to be a quasi-pseudo- b-metric. We illustrate the process of embedding reducible fuzzy graphs into an embedding space that is a normed space ℓ_∞. The primary aim is to explicate how such b-metric structures and their embeddings can be implemented in examples of $RD_{\mathcal{FG}}(V,E)$ or in samples from various real-life problems modeled on reducible fuzzy graphs.

The structure of this paper proceeds as follows: Initially, in Section 2, we introduce some concepts and facts about fuzzy graph theory, along with the basic concepts of b-metricmspaces and normed spaces. In Section 3, a modern method for depicting a reducible fuzzy digraph, based on the b_{RD}-distance function, is presented. In fact, this section is dedicated to discussing digraphs that give rise to a particular kind of interesting b-metric space. We prove that the b_{RD}-distance function is a quasi-pseudo- b-metric, which leads to embedding such a b-metric space into a specified normed space ℓ_∞. Additionally, pivotal outcomes are illustrated through a numerical example and a real-world application in this section.

2. Fundamental concepts and methods

2.1. Core concepts of fuzzy digraphs

In this study, the term "digraph" alludes to a directed graph without self-loops (edges relating a vertex to itself) or multiple arcs (parallel edges relating one vertex to another). This subsection offers some principal concepts and properties related to the research, starting with the main terminologies of a directed graph. A directed graph $D_G(V,E)$ is a specific relation $E \subset V \times V$ between the points V representing nodes (or vertices) and a set E of directed edges (or directed arcs). Each directed edge in E is a pair of distinct nodes. A directed path, denoted as P, in $D_G(V,E)$ is a sequence of distinguished vertices and directed edges linking the vertices in the specified direction [26]. In addition, we outline the essential graphical concepts of irreducible and reducible digraph as follows.

Definition 2.1. [22] $D_G(V,E)$ is called strongly connected (also called irreducible) if it has a directed path from v_i to v_j and a directed path from v_j to v_i, for every pair of nodes v_i and v_j belonging to V, see Fig. 1, labeled as (a).

Definition 2.2. [23] A digraph $D_G(V,E)$ is considered not strongly connected, i.e., it is reducible or, in short, a reducible digraph, if there is no directed path between at least one pair of vertices, see Fig. 1, labeled as (b) and (c).

In an algebraic method, as illustrated in the introduction, each digraph is reducible if and only if its adjacency matrix B is reducible. In a meaningful or theoretical graph, as mentioned in Definition 2.2, each digraph is reducible if there exist vertices v_i and v_j such that there is no directed path traveling from one vertex like v_i to another like v_j. Observe that in Fig. 1(b), there is no directed path traveling from vertex v_3 to another vertex such as v_1 (or v_2). Similarly, in Fig. 1(c), there is no directed path traveling from vertex like v_1 (or v_2) to vertex v_3. The following definitions pertain to concepts in the fuzziness of directed graphs.

Definition 2.3. [15] A fuzzy graph $D_{\mathcal{FG}}(\sigma, \mu)$ is defined by the mappings $\sigma : V \rightarrow [0,1]$ and $\mu : V \times V \rightarrow [0,1]$, where V is the underlying vertex set and E is the underlying edge set. The mapping σ represents a fuzzy subset of V, and μ represents a fuzzy relation on σ, satisfying $\mu\left(v_i, v_j\right) \leq \sigma\left(v_i\right) \wedge \sigma\left(v_j\right), \forall v_i, v_j \in V$, where $\left(v_i, v_j\right)$ denotes the directed edge between v_i and v_j. In this context, $\sigma\left(v_i\right) \wedge \sigma\left(v_j\right)$ signifies the minimum of $\sigma\left(v_i\right)$ and $\sigma\left(v_j\right)$. The term σ denotes the fuzzy vertex set of V, and μ denotes the fuzzy edge set of E. The underlying structure of a fuzzy digraph $D_{\mathcal{FG}}(\sigma, \mu)$ typically corresponds to $D_G(V, E)$.

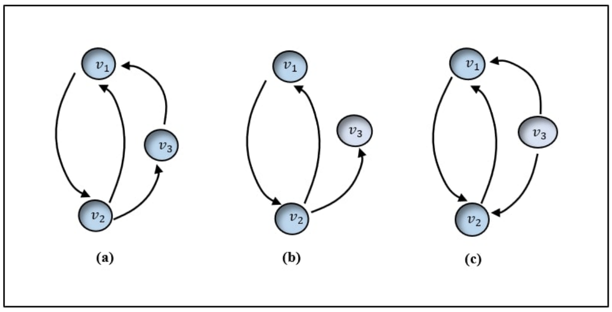

Fig. 1. (a) Irreducible digraph; (b) and (c) Reducible digraphs.

If $D_{\mathcal{FG}}(\sigma, \mu)$ is a fuzzy graph, then some key concepts are presented as follows [15].

(1) A fuzzy graph $D_{\mathcal{FG}G_1}(\rho, \omega)$ is considered a fuzzy subgraph of $D_{\mathcal{FG}}(\sigma, \mu)$ if $\rho\left(v_i\right) \leq \sigma\left(v_i\right), \forall v_i \in V$ and $\omega\left(v_i, v_j\right) \leq \mu\left(v_i, v_j\right), \forall v_i, v_j \in V$.

(2) A sequence $P : v_i = v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k, \ldots, v_{n-1}, e_n, v_n = v_j$ is defined as a directed path from a vertex v_i to a vertex v_j in $D_{\mathcal{FG}}(\sigma, \mu)$. The sequence P consists of distinct vertices and edges starting from v_i and ending at v_j, with the condition that the membership value $\mu\left(\left(v_{k-1}, v_k\right)\right) > 0$ for $k = 1, \cdots, n$.

(3) The number of edges, indicated by n, is commonly referred to as the usual length of the path P.

(4) If vertices v_i and v_j coincide in a directed path P, then P is referred to as a cycle.

The fuzziness of graphs and their generalizations has gained widespread use in multiple fields, demonstrating their versatility in modeling and decision-making in imprecise contexts. For example, in healthcare, energy-based bipolar intuitionistic fuzzy digraphs and bipolar fuzzy p-competition graphs have been employed to prioritize COVID-19 vaccines, offering robust frameworks for handling imprecise and conflicting information [27, 28]. Likewise, in disaster management, bipolar intuitionistic fuzzy graphs have been utilized to identify flood-vulnerable regions, while split domination in picture fuzzy graphs has been proposed for optimizing disaster response strategies [29, 30]. In crime analysis studies, the development of the F-index for fuzzy graphs has provided a novel approach to analyzing Indian railway crime data [31]. Additionally, the application of edge F-index in molecular chemistry has revealed the potential of fuzzy graphs in modeling complex molecular structures [32]. In spite of this progress, challenges such as computational complexity, reducibility, and asymmetry persist in fuzzy digraphs. This study strives to mitigate these concerns by proposing a low-distortion $\mathcal{D}$-embedding of the b-metric space of fuzzy digraphs into a specific normed space.

2.2. Key distances and their metric characteristics

In a fuzzy graph, the distances between vertices v_i and v_j often represent the length of the shortest path between them (see e.g., [15, 33, 34]). For alternative distance notions in fuzzy graphs, one may refer to [35–37]. Various distance metric functions can be applied based on the certain problem at hand. This interpretation suggests that a measurement of distance could be symbolic (non-physical) rather than tangible. However, this work also delves into numerous generalizations of metric spaces. For instance, we show the b-metric space, quasi-metric space, quasi- b-metric space, quasi-pseudo-metric space, and quasi-pseudo- b-metric space (see [38–41]). Their notions are as follow:

Definition 2.4. Let X be a nonempty set and $M \geq 1$ a real number. Let $d_b : \mathrm{X} \times \mathrm{X} \rightarrow [0, \infty)$ be a mapping and $x, y, z \in \mathrm{X}$:

(d_{b1}) $d_b(x, y) = 0$ if and only if $x = y$

(d_{b2}) $d_b(x, x) = 0$ (but perhaps $d_b(x, y) = 0$ for some distinct points $x \neq y$)

(d_{b3}) $d_b(x, y) = d_b(y, x)$

(d_{b4}) $d_b(x, z) \leq d_b(x, y) + d_b(y, z)$

(d_{b5}) $\quad d_b(x,z) \le M\left[d_b(x,y) + d_b(y,z)\right], M \ge 1.$

Then:

(1) (X, d_b) is called a metric space if the axioms d_{b1}, d_{b3} and d_{b4} are satisfied.

(2) (X, d_b) is called a b-metric space if the axioms d_{b1}, d_{b3} and d_{b5} are satisfied.

(3) (X, d_b) is called a pseudo-metric space if the axioms d_{b2}, d_{b3} and d_{b4} are satisfied.

(4) (X, d_b) is called a pseudo- b-metric space if the axioms d_{b2}, d_{b3} and d_{b5} are satisfied.

(5) (X, d_b) is called a quasi-metric space if the axioms d_{b1} and d_{b4} are satisfied.

(6) (X, d_b) is called a quasi- b-metric space if the axioms d_{b1} and d_{b5} are satisfied.

(7) (X, d_b) is called a quasi-pseudo-metric space if the axioms d_{b2} and d_{b4} are satisfied.

(8) (X, d_b) is called a quasi-pseudo- b-metric space if the axioms d_{b2} and d_{b5} are satisfied.

From Definition 2.4, the following relations are clearly fulfilled:

i. metric space (1) $\Longrightarrow$ quasi-metric space (5) $\Longrightarrow$ quasi-pseudo-metric space (7).

ii. metric space (1) $\Longrightarrow$ b-metric space (2) $\Longrightarrow$ quasi- b-metric space (6) $\Longrightarrow$ quasi-pseudo- b-metric space (8).

However, the converse directions are not always correct [39–41]. The explicitness of the definition of the quasi-pseudo- b-metric [42] lies in its generalization of metric spaces, where the conventional triangle inequality is replaced by a weaker condition (see axiom d_{b5}), often involving a constant $M \ge 1$ and a relaxation of the symmetry or positivity conditions (see axiom d_{b2}). To illustrate this distinction, consider the set $X = \mathbb{R}$ equipped with the function $d_b : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by $d_b(x,y) = \begin{cases} 0 & \text{if } x \le y \\ |x-y| & \text{if } x > y \end{cases}$, where $\mathbb{R}$ denotes the real numbers. This function d_b satisfies the axioms of a quasi-pseudo-b-metric space with $M = 1$. Specifically, d_b is non-negative, satisfies the weak identity condition $(d_b(x,x) = 0)$, and is asymmetric ($d_b(x,y) \ne d_b(y,x)$ unless $x = y$). Furthermore, d_b fulfills the relaxed triangle inequality $(d_b(x,z) \le M\left[d_b(x,y) + d_b(y,z)\right])$ for all $x, y, z \in \mathbb{R}$. However, d_b does not satisfy the identity of indiscernible, that is, $d_b(x,y) = 0$ does not imply $x = y$. This is evidenced by the case $d_b(1,2) = 0$ where $1 \ne 2$. Consequently, d_b does not satisfy the axioms of a quasi-b-metric space, as it fails to meet the stricter requirement of the identity of indiscernible.

2.3. Core Principles of Metric Space Embeddability

Several researchers have examined distinct types of $\mathcal{D}$-embeddability of metric spaces into numerous normed spaces (see, for instance [43, 44]). While much of previous research has focused on the embeddings of symmetric metric spaces and standard graphs [45–47], the exploration of quasi-pseudo- b-metric spaces linked to reducible digraphs remains underdeveloped, as shown in Table 1. This study bridges this gap by introducing a comprehensive framework for constructing embeddings that preserve essential metric properties while enabling dimensionality reduction. By addressing the unique structural challenges of reducible digraphs, the study advances embedding theories and fills a critical theoretical gap in graph structure analysis.

To reinforce our research framework, we provide the primary notions within this context that align with our research objectives. Here the ℓ_∞ norm, or maximum norm, is given by $\|x\|_\infty = \max_i \{|x_i|\}$, for a point $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$, where $\mathbb{R}$ points to the real numbers [43]. We denote by ℓ_∞^k the normed space $\left(\mathbb{R}^k, \|x\|_\infty\right)$ that is considered in our study. We employ embedding into ℓ_∞ as a convenient shorthand for embedding into ℓ_∞^k for various positive integer values of k. We devote our interest to overarching embedding techniques as delineated in the following context.

Definition 2.5. [48] Let $\mathcal{V}$ be a metric space with metric p and let $\mathcal{X}$ be a normed space (whose norm is usually indicated by $\|\|$). Suppose $f : \mathcal{V} \to \mathcal{X}$ be a mapping. f is defined as a $\mathcal{D}$-embedding (or a mapping with distortion at most $\mathcal{D}$, where $\mathcal{D} \ge 0$ is a real number), if it fulfills $\frac{1}{\mathcal{D}} p(v,u) \le \|f(v) - f(u)\| \le p(v,u)$ for any two points $v, u \in \mathcal{V}$. We say that $\mathcal{V}\mathcal{D}$-embeds into X if there exists a $\mathcal{D}$-embedding $f : \mathcal{V} \to \mathcal{X}$.

Distortion is defined as the lowest value, $\mathcal{D}$, that ensures the given inequality above. Broadly, we can distinguish the embedding methods into two major categories: isometric embeddings [49], which are embeddings with distortion 1, and general embedding methods as given in Definition 2.5. Our objective is to discover embeddings with minimal distortion. Therefore, $\mathcal{D}$-embeddings can prove useful for representing reducible fuzzy digraphs and quasi-pseudo- b-metric spaces, aiding in the visualization of their structures, as demonstrated in this study.

Table 1. Development of the theoretical concepts (Recent Graph Embedding Works).

Year	Author (s)	Title of work	Key contributions
2019	A. Gu, et al.	Learning mixed-curvature representations in product spaces [45].	An embedding framework using a product manifold of Spherical, Hyperbolic, and Euclidean spaces.
2021	F. Lopez, et al.	Symmetric spaces for graph embeddings: A finsler riemannian approach [46].	A framework for graph embeddings using symmetric spaces and Riemannian optimization.
2022	F. Di Giovanni, et al.	Heterogeneous manifolds for curvature-aware graph embedding [47].	A framework for graph embeddings using heterogeneous rotationally-symmetric manifolds to capture local curvature variations.
2024	D. Taha, et al.	Normed Spaces for Graph Embedding [44].	Unlike Riemannian manifolds (hyperbolic, spherical), normed spaces (Euclidean space with ℓ_2) provide a simpler, more general framework for embedding metric spaces of a graph.

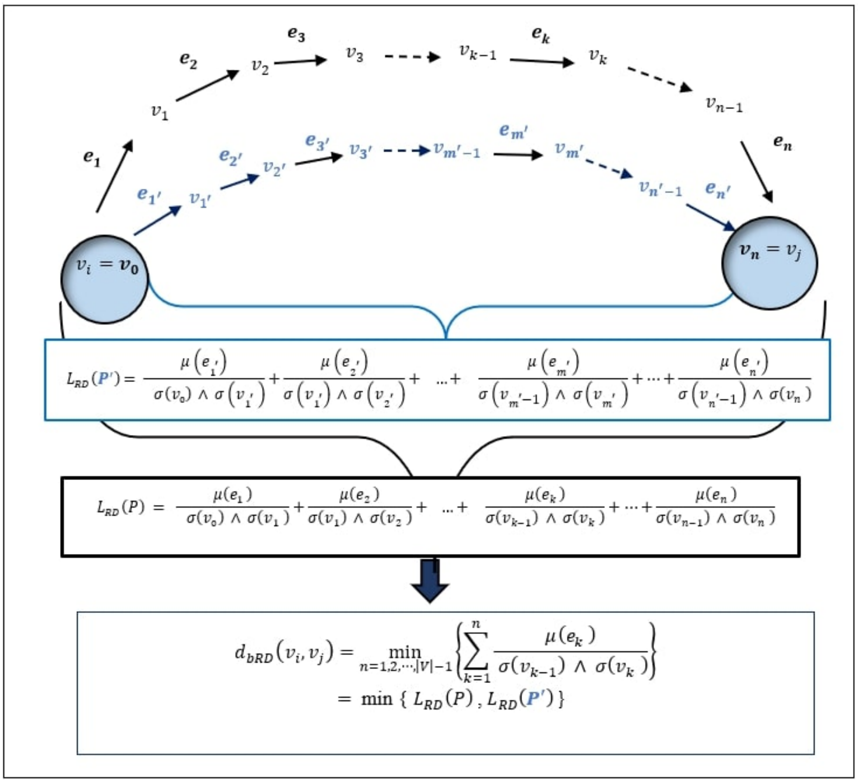

Fig. 2. Illustration of b_{RD}-distance $\left(d_{bRD}\left(v_i, v_j\right)\right)$ between two different vertices $v_i = v_0$ and $v_j = v_n$ in $RD_{\mathcal{FG}}(V, E)$.

3. Results and discussion

3.1. b_{RD}-Metric on Reducible Fuzzy Digraph

In this section, employing the membership values of vertices, represented by $\sigma(v)$, and edges, represented by $\mu(e)$, we introduce a new concept of the RD-length, denoted as the length of any path in a reducible fuzzy digraph $RD_{\mathcal{FG}}(V, E)$. Then, by using this RD-length of a path, we define the b_{RD}-distance function between any two different vertices in $RD_{\mathcal{FG}}(V, E)$ and demonstrate its properties as a quasi-pseudo- b-metric. This guides the formulation of an important definition, representing the required new structure of $RD_{\mathcal{FG}}(V, E)$ as will be shown in Definition 3.5.

Definition 3.1. RD-length of a path in $RD_{\mathcal{FG}}(V,E)$. Let $RD_{\mathcal{FG}}(V,E)$ be a directed graph representing a reducible fuzzy graph. A path $P_{e_1,e_2,\cdots,e_k,\cdots,e_n}$ in $RD_{\mathcal{FG}}(V,E)$ is a sequence of dissimilar vertices and edges in $RD_{\mathcal{FG}}(V,E)$ that begins at v_i and ends at v_j, where the membership value $\mu((v_{k-1},v_k)) = \mu(e_k) > 0$, for $1 \le k \le n$, with each edge $e_k \in P_{e_1,e_2,\cdots,e_k,\cdots,e_n}$ and n representing the whole number of edges in this path. Then, a RD-length L_{RD} of a path $P_{e_1,e_2,\cdots,e_k,\cdots,e_n}$ in $RD_{\mathcal{FG}}(V,E)$ is defined as

$$L_{RD}(P) = \sum_{k=1}^{n} \frac{\mu((v_{k-1},v_k))}{\sigma(v_{k-1}) \wedge \sigma(v_k)} = \sum_{k=1}^{n} \frac{\mu(e_k)}{\sigma(v_{k-1}) \wedge \sigma(v_k)} = \frac{\mu(e_1)}{\sigma(v_0) \wedge \sigma(v_1)} + \frac{\mu(e_2)}{\sigma(v_1) \wedge \sigma(v_2)} + \dots + \frac{\mu(e_k)}{\sigma(v_{k-1}) \wedge \sigma(v_k)} + \cdots + \frac{\mu(e_n)}{\sigma(v_{n-1}) \wedge \sigma(v_n)}$$

(see Fig. 2).

Definition 3.2. b_{RD}-distance in $RD_{\mathcal{FG}}(V,E)$. Let n be the whole number of edges in the path $P_{e_1,e_2,\cdots,e_n}$ and $|V|$ is the number of all vertices in $RD_{\mathcal{FG}}(V,E)$. Let $L_{RD}(P)$ be a RD-length of any path that initiated by v_i and terminated by v_j and given by $L_{RD}(P) = \sum_{k=1}^{n} \frac{\mu(e_k)}{\sigma(v_{k-1})\wedge\sigma(v_k)}$. Then, the b_{RD}-distance, denoted by $d_{bRD}\left(v_i,v_j\right)$, between two different vertices v_i and v_j in $RD_{\mathcal{FG}}(V,E)$ is determined by

$$d_{bRD}\left(v_i,v_j\right) = \begin{cases} \min_{n=1,2,\cdots,|V|-1} \left\{\sum_{k=1}^{n} \frac{\mu(e_k)}{\sigma(v_{k-1})\wedge\sigma(v_k)}\right\} & \text{if } i \neq j \text{ and for each edge } e_k \in P_{e_1,e_2,\cdots,e_n} \\ 0 & \text{if } i = j \text{ or } \sigma(v_{k-1}) \wedge \sigma(v_k) = 0 \text{ or } e_k \notin E \end{cases} \tag{1}$$

(see Fig. 2).

In Definition 3.2, note that all vertices in a path $P_{e_1,e_2,\cdots,e_k,\cdots,e_n}$ must be dissimilar. Therefore, the minimum RD-length of a path in Eq. (1) is calculated over all likely paths that link v_i and v_j (specifically, for n ranging from 1 to $|V|-1$).

Remark 3.3. If the membership value $\mu(e_k)$ between two vertices v_{k-1} and v_k in Definition 3.2 is $\mu((v_{k-1},v_k)) = \mu(e_k) > 0$ for any edge $e_k \in E$ in $RD_{\mathcal{FG}}(V,E)$, then, there is a link from v_{k-1} to v_k, say $\mu(e_k)$, it is not required to assign equivalent membership value as a link from v_k to v_{k-1}, say $\mu(e_{k-1})$, i.e., $\mu(e_k) \neq \mu(e_{k-1})$, wherefore the b_{RD}-distance function defined in Eq. (1) clearly satisfies $d_{bRD}\left(v_i,v_j\right) \neq d_{bRD}\left(v_j,v_i\right)$ for all $v_i, v_j \in V$.

Thus, the emergence of this b_{RD}-distance prompts a common question: what types of distances can be classified under metric structures. In other words, what is the classification of digraphs that can be constructed using b-metric structures. This question is answered by the subsequent theorem.

Theorem 3.4. Let $RD_{\mathcal{FG}}(V,E)$ be a digraph of a reducible fuzzy graph. Then, the b_{RD}-distance, d_{bRD} as defined in Eq. (1) between every pair of different vertices in $RD_{\mathcal{FG}}(V,E)$, is a quasi-pseudo- b-metric.

Proof:

Given that the digraph is reducible, there exist at least two separate vertices that cannot be linked by any path, whereas any other pair of vertices can be joined by a path or several paths. Therefore, by b_{RD}-distance function, d_{bRD} defined by Eq. (1), it is clear that $d_{bRD}\left(v_i,v_j\right) \geq 0, \forall v_i, v_j \in V$, in addition to $d_{bRD}\left(v_i,v_j\right) \neq d_{bRD}\left(v_j,v_i\right)$ comes from Remark 3.3.

Thereafter, it can be observed that the axiom d_{b2} of Definition 2.4 holds for all pairs of vertices in $RD_{\mathcal{FG}}(V,E)$. To prove d_{b2}, assume that $v_i = v_j$, then, by the function d_{bRD} given as in Eq. (1), $d_{bRD}(v_i,v_i) = 0$, but possibly $d_{bRD}\left(v_i,v_j\right) = 0$, for some distinct vertices $v_i \neq v_j$ (due to the fact that the graph $RD_{\mathcal{FG}}(V,E)$ is reducible).

Then, It suffices to show that the axiom d_{b5} of Definition 2.4 holds by mathematical induction as follows:

Let $d_{bRD}\left(v_i,v_j\right) = c_i > 0$ for all $i \neq j = 1,2,3,\ldots,n$, and for all $e_i \in P_{e_1,e_2,\cdots,e_n}$ then, we need to prove that

$$d_{bRD}(v_1,v_n) \leq M[d_{bRD}(v_1,v_2) + d_{bRD}(v_2,v_3) + \cdots + d_{bRD}(v_{n-1},v_n)], \text{ where } M \geq 1.$$

Now, an initial step is as follows:

S_1 : if $1 = i \neq j = 2$, then $d_{bRD}(v_1,v_2) = d_{bRD}(v_1,v_2) = c_1$. So ,it is readily true for $n = 1$.

Our hypothesis asserts that

$S_n : d_{bRD}(v_1,v_k) \leq M[d_{bRD}(v_1,v_2) + d_{bRD}(v_2,v_3) + \cdots + d_{bRD}(v_{k-1},v_k)]$ is true for $n = k$. Then, we should prove that the result is true for $n = k+1$. This means $S_{n+1} : d_{bRD}(v_1,v_{k+1}) \leq M[d_{bRD}(v_1,v_2) + d_{bRD}(v_2,v_3) + \cdots + d_{bRD}(v_k,v_{k+1})]$ must be true. Getting to step S_{n+1}, note that

$$
\begin{aligned}
&[d_{bRD}(v_1,v_2)+d_{bRD}(v_2,v_3)+\cdots+d_{bRD}(v_{k-1},v_k)+d_{bRD}(v_k,v_{k+1})] \\
&= M[d_{bRD}(v_1,v_2)+d_{bRD}(v_2,v_3)+\cdots+d_{bRD}(v_{k-1},v_k)]+M[d_{bRD}(v_k,v_{k+1})] \\
&\geq d_{bRD}(v_1,v_k)+M[d_{bRD}(v_k,v_{k+1})] \quad (\text{ by } S_n) \\
&= d_{bRD}(v_1,v_k)-M[d_{bRD}(v_1,v_k)]+M[d_{bRD}(v_1,v_k)]+M[d_{bRD}(v_k,v_{k+1})] \\
&= d_{bRD}(v_1,v_k)-M[d_{bRD}(v_1,v_k)]+M[d_{bRD}(v_1,v_k)+d_{bRD}(v_k,v_{k+1})] \\
&\geq d_{bRD}(v_1,v_k)-M[d_{bRD}(v_1,v_k)]+d_{bRD}(v_1,v_{k+1}) \text{ (by our hypothesis)} \\
&= c_k-Mc_k+d_{bRD}(v_1,v_{k+1}) \text{ (by taking } d_{MRD}(v_1,v_k)=c_k>0) \\
&= (1-M)c_k+d_{bRD}(v_1,v_{k+1}) \\
&\geq d_{bRD}(v_1,v_{k+1}) \quad (\text{ since if } M=1, \text{ then } (1-M)c_k=0, \text{ and if } M>1, \text{ then} \\
&(1-M)c_k<0), \text{ as required.}
\end{aligned}
$$

Thus, the result is true for all $i \neq j \neq k = 1,2,3,\ldots,n$, which means the axiom d_{b5} of the function d_{bRD} is fulfilled, i.e., $d_{bRD}\left(v_i,v_j\right) \leq M\left[d_{bRD}(v_i,v_k)+d_{bRD}\left(v_k,v_j\right)\right]$ for all $v_i,v_j,v_k \in V$.

Then, by Definition 2.4,(8), d_{bRD} is a quasi-pseudo- b-metric. This concludes the proof. Theorem 3.4 reveals that b_{RD}-distance, known as d_{bRD}, in any reducible fuzzy digraph $RD_{\mathcal{FG}}(V,E)$ forms a quasi-pseudo- b-metric. As a result, the b_{RD}-distance plays a fundamental role in establishing quasi-pseudo-b-metric spaces for fuzzy digraphs. Its distinctive properties - asymmetry, path-dependence, and fuzzy edge membership - naturally align with the defining features of quasi-pseudo-b-metrics. The b_{RD}-distance satisfies the essential axioms of non-negativity, pseudo-identity of indiscernible, and the relaxed triangle inequality, making it a suitable candidate for this space, as formalized in Theorem 3.4. Consequently, this theorem guides the formulation of the following significant definition, representing the desired new structure of $RD_{\mathcal{FG}}(V,E)$.

Definition 3.5. Quasi-pseudo- b-metric space of $RD_{\mathcal{FG}}(V,E)$. Let $RD_{\mathcal{FG}}(V,E)$ be a digraph of a reducible fuzzy graph. The b_{RD}-distance function d_{bRD} between any two different vertices v_i,v_j in $RD_{\mathcal{FG}}(V,E)$ defined as

$$
\begin{aligned}
&d_{bRD}\left(v_i,v_j\right) \\
&= \begin{cases} \min_{n=1,2,\cdots,|V|-1}\left\{\sum_{k=1}^{n}\frac{\mu(e_k)}{\sigma(v_{k-1})\wedge\sigma(v_k)}\right\} & \text{if } i \neq j \text{ and for each edge } e_k \in P_{e_1,e_2,\cdots,e_n} \\ 0 & \text{if } i=j \text{ or } \sigma(v_{k-1})\wedge\sigma(v_k)=0 \text{ or } e_k \notin E \end{cases}
\end{aligned}
$$

is called a quasi-pseudo- b-metric of $RD_{\mathcal{FG}}(V,E)$ which holds.

1. $d_{\mathrm{bRD}}(v_i,v_i)=0$
2. $d_{bRD}\left(v_i,v_j\right) \leq M\left[d_{bRD}(v_i,v_k)+d_{bRD}\left(v_k,v_j\right)\right]$, $\forall v_i,v_j,v_k \in V$, and a constant M

is the minimum possible value that cannot be replaced by a smaller positive real number satisfying all cases in inequality 2.

A triblet (V,d_{bRD},M) is referred to as a quasi-pseudo- b-metric space of $RD_{\mathcal{FG}}(V,E)$ if d_{bRD} satisfies the postulates 1 and 2.

3.2. Embedding b_{RD}-Metric of Reducible Fuzzy Digraph into Normed Space

In this section, we present a method for constructing low-distortion $\mathcal{D}$-embedding of a quasi-pseudo- b-metric space of a reducible fuzzy graph into space ℓ_∞. It is worth to looking at the digraph being reducible and M is at least 1 which represents the lowest possible value that satisfies all cases in inequality 2 in Definition 3.5, hence, we consider $\mathcal{D}=3M$ that is greater than or equal to 3 . Subsequently, the next concept is introduced to illustrate this technique for generating $\mathcal{D}$-embedding method that produce a mapping based on the b_{RD}-distance function d_{bRD}.

Definition 3.6. Define $V(R)$ as the set of vertices that cannot be reached by a path initiated (or terminated) by these vertices in a digraph $RD_{\mathcal{FG}}(V,E)$. In other words, $V(R)$ is the set of vertices v_m which satisfies $d_{bRD}(v_m,v_i)=0$(or $d_{bRD}(v_i,v_m)=0$), for all $v_i \neq v_m$ and $v_i,v_m \in V$.

Since $V(R)$ represents a set of vertices that cannot be linked by a path starting or ending within this set, removing $V(R)$ is necessary to achieve a low-distortion $\mathcal{D}$-embedding in a digraph $RD_{\mathcal{FG}}(V,E)$ as shown in the following theorem.

Theorem 3.7. Let $\mathcal{D} = 3M$ be given, let (V, d_{bRD}, M) be a quasi-pseudo- b-metric space of a reducible fuzzy graph $RD_{\mathcal{FG}}(V, E)$. Then there is a $\mathcal{D}$-embedding of V into normed space ℓ_∞^k with $k = |V| - |V(R)|$, where $|V(R)|$ is the number of vertices in $V(R)$ that are removed during dimensionality reduction of the embedding.

Proof:

To assign a $\mathcal{D}$-embedding $f : V \to \ell_\infty^k$ means to define k mappings $f_1, f_2, \ldots, f_k : V \to \mathbb{R}$, representing the coordinates of the embedded vertices, where $\mathbb{R}$ denotes the real numbers.

Here, the dimensionality of the embedding's image f is reduced by removing the coordinates corresponding to the vertices that belong to $V(R)$. Consequently, the coordinates in ℓ_∞^k are indexed by the vertices of V except those in $V(R)$, i.e., $V/V(R)$. Then, one of the main ideas in building of such a $\mathcal{D}$-embedding f is to define each f_i as the distance to an appropriate subset $A_i \subseteq V/V(R)$. That is, the i th coordinate in ℓ_∞^k is given by $f_i(v) = d_{bRD}(v, A_i) = \min_{u \in A_i} \{d_{bRD}(v, u)\}$ for all $i = 1, 2, \ldots, k$ and all $v, u \in V/V(R)$, where we choose a subset $A_i = \{u \in V/V(R), d_{bRD}(v, u) \neq 0\}$ for all pairs $\{v, u\}$ in $V/V(R)$.

To show that f is a $\mathcal{D}$-embedding, without loss of generality, we can require it to be nonexpanding. Thus, the first goal is to establish the following inequality:

$$|f_i(v) - f_i(u)| \leq d_{bRD}(v, u) \quad \text{for all } i = 1, 2, \ldots, k \quad \text{and all } v, u \in V/V(R) \tag{2}$$

This means it must prove the inequality

$$|f_i(v) - f_i(u)| = |d_{bRD}(v, A_i) - d_{bRD}(u, A_i)| = \left| \min_{u \in A_i} \{d_{bRD}(v, u)\} - \min_{v \in A_i} \{d_{bRD}(u, v)\} \right| \leq d_{bRD}(v, u)$$

, for any $v, u \in V/V(R)$ and A_i is a fixed set associated with the embedding f_i for all $i = 1, 2, \ldots, k$. Then, it can precisely be analyzed two cases based on the values of $d_{bRD}(v, A_i)$ and $d_{bRD}(u, A_i)$ as follows:

Case 1: if $d_{bRD}(v, A_i) \geq d_{bRD}(u, A_i)$:

In this case, $d_{bRD}(v, A_i) - d_{bRD}(u, A_i) \geq 0$, then, by our definition f_i and since $d_{bRD}(v, u)$ is greater than or equal to $d_{bRD}(v, A_i)$, it ensures that the difference between the minimum distances to the same point A_i from v and u is bounded by the distance between v and u, i.e., $d_{bRD}(v, A_i) - d_{bRD}(u, A_i) \leq d_{bRD}(v, u)$. This directly implies $|d_{bRD}(v, A_i) - d_{bRD}(u, A_i)| \leq d_{bRD}(v, u)$.

Case 2 : if $d_{\mathrm{bRD}}(v, A_i) \leq d_{\mathrm{bRD}}(u, A_i)$:

In this case, $d_{bRD}(u, A_i) - d_{bRD}(v, A_i) \geq 0$, and in same behavior as case 1 , we have $d_{bRD}(u, A_i) - d_{bRD}(v, A_i) \leq d_{bRD}(u, v)$ This directly yields $|d_{bRD}(u, A_i) - d_{bRD}(v, A_i)| \leq d_{bRD}(u, v)$.

Therefore, we have successfully proved the inequality (2). Thus, since each f_i is nonexpanding, f is nonexpanding as well. The second goal is to prove the $\mathcal{D}$-embedding condition that means for any pair $\{v, u\}$ of vertices of $V/V(R)$, there is a coordinate $i = i(v, u)$ that satisfies the following inequality:

$$|f_i(v) - f_i(u)| \geq \frac{1}{\mathcal{D}} d_{bRD}(v, u) \quad \text{for all } i = 1, 2, \ldots, k \tag{3}$$

Then, it is clear that the choice of the coordinates f_i and the sets A_i, defining the $\mathcal{D}$-embedding as explained above, ensures the existence of a vertex $v \in V/V(R)$ with $d_{bRD}(v, u) \neq 0$ for all pairs $\{v, u\}$ belonging to $V/V(R)$. So, let us consider a fixed pair $\{v, u\}$ and select the suitable index i as $i = i(v, u)$. It remains to prove the inequality (3). Now, observe that $\mathcal{D} = 3M$ is greater than or equal to 3 (since M is at least 1). Therefore, we can express the inequality (3) as follows:

$$|f_i(v) - f_i(u)| \geq \frac{1}{3M} d_{bRD}(v, u), \text{ for all } i = 1, 2, \ldots, k \tag{4}$$

Thus, two cases are considered. In case $f_i(v) > f_i(u)$:

Since $|f_i(v) - f_i(u)| = \left|\min_{u \in A_i} \{d_{bRD}(v, u)\} - \min_{v \in A_i} \{d_{bRD}(u, v)\}\right|$, then we can choose z such that $f_i(v) = d_{bRD}(v, z)$ and by definition of A_i, we know that $d_{bRD}(v, z) \neq 0$ is non-zero for the respective vertex z. Also, let's choose w such that $f_i(u) = d_{bRD}(u, w)$ and $d_{bRD}(u, w) \neq 0$ is non-zero for the respective vertex w.

It is obvious that $d_{bRD}(v, u) < \mathcal{D} \cdot d_{bRD}(v, u) = (3M) \cdot d_{bRD}(v, u) = (-3M)(-d_{bRD}(v, u)) <$ (-3 M) $[d_{bRD}(u, w) - d_{bRD}(v, z)]$ [since we have inequality (2) and $f_i(v) = d_{bRD}(v, z) > d_{bRD}(u, w) = f_i(u)$, then we get $f_i(v) - f_i(u) = d_{bRD}(v, z) - d_{bRD}(u, w) < d_{bRD}(v, u)$, and thus,

$$-d_{bRD}(v,z) + d_{bRD}(u,w) > -d_{bRD}(v,u)]$$
$$= (-3M)d_{bRD}(u,w) + (3M)d_{bRD}(v,z)$$
$$= (3M)d_{bRD}(v,z) - (3M)d_{bRD}(u,w)$$
$$= (3M)\left[d_{bRD}(v,z) - d_{bRD}(u,w)\right]$$

Therefore, $\frac{1}{3M}d_{bRD}(v,u) < [d_{bRD}(v,z) - d_{bRD}(u,w)] = [f_i(v) - f_i(u)]$.

The proof is similar if in case $f_i(u) > f_i(v)$ and it yields $\frac{1}{3M}d_{bRD}(u,v) < [f_i(u) - f_i(v)]$.

This completes the proof of the inequality (4) for both cases and thus, it finishes the proof of Theorem 3.7.

The b-metric framework enhances the embedding of large data entities or nodes into low-dimensional spaces by introducing a relaxation parameter M, as defined in Definition 3.5, which allows for controlled distortions in distance constraints. A key methodological advancement, as outlined in this Subsection, is the systematic removal of vertices that cannot be linked by a path starting or ending within a specified set, denoted as $V(R)$. This process, as demonstrated in Theorem 3.7, reduces the graph's structural complexity, enabling low-distortion embeddings while preserving essential structural properties. By facilitating dimensionality reduction and reducing computational complexity, the b-metric framework supports the efficient embedding of large, irregular, and noisy datasets, making it a robust tool for dimensionality reduction, graph embeddings, and large-scale data visualization.

This methodology fills a pivotal gap in graph theory by providing a robust framework for analyzing complex, not strongly connected structures that traditional methods, such as adjacency matrices, cannot adequately capture. By modeling asymmetric distances and fuzzy path structures inherent in reducible digraphs, the framework advances embedding theories and supplies practical tools for applications characterized by reducibility, asymmetry, and vagueness in fuzzy digraphs.

3.3. Application of the Research

3.3.1. Numerical Example of Application

In this subsection, we present a detailed example of a reducible fuzzy digraph $RD_{\mathcal{FG}}(V,E)$ and discuss some interesting findings that have been derived from the previous sections of this research. Fig. 3 shows an example of such a digraph composed of the set of vertices $V = \{v_o, v_1, v_2, v_3, v_4\}$ together with membership values $\sigma(v_o) = 0.5, \sigma(v_1) = 0.4$, $\sigma(v_2) = 0.6, \sigma(v_3) = 0.3, \sigma(v_4) = 0.4$ and the set of edges $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ together with the membership values of edges $\mu((v_o, v_1)) = \mu(e_1) = 0.2$, $\mu((v_1, v_2)) = \mu(e_2) = 0.35, \mu((v_2, v_3)) = \mu(e_3) = 0.2, \mu((v_3, v_o)) = \mu(e_4) = 0.3$, $\mu((v_4, v_o)) = \mu(e_5) = 0.2, \mu((v_4, v_3)) = \mu(e_6) = 0.25$, which is proffered in Fig. 3.

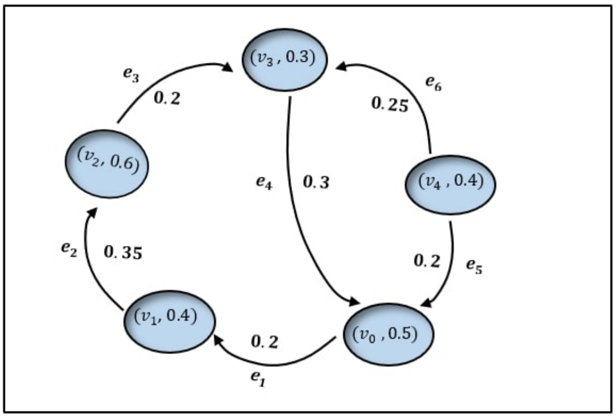

Fig. 3. Reducible Fuzzy Digraph $RD_{\mathcal{FG}}(V,E)$ with $|V| = 5$ and $|E| = 6$.

Thus, the b_{RD}-distances $d_{bRD}\left(v_i, v_j\right)$ that initiated by v_i and ended by v_j can be computed by the function d_{bRD} represented by Eq. (1) with the following details.

The all possible paths that connect v_o to other vertices in Fig. 3 are as follows:

1. The possible paths that link v_o to v_1 are only a path P_{e_1} of length one $v_o \xrightarrow{e_1} v_1$, then the minimum RD-length of this path is $d_{bRD}(v_o, v_1) = \frac{\mu(e_1)}{\sigma(v_0)\wedge\sigma(v_1)} = \frac{0.2}{0.5\wedge 0.4} = \frac{0.2}{0.4} = 0.5$.
2. The possible paths that link v_o to v_2 are only a path P_{e_1,e_2} of length two $v_o \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2$, then the minimum RD-length of this path is $d_{bRD}(v_o, v_2) = \frac{\mu(e_1)}{\sigma(v_0)\wedge\sigma(v_1)} + \frac{\mu(e_2)}{\sigma(v_1)\wedge\sigma(v_2)} = \frac{0.2}{0.4} + \frac{0.35}{0.4} = 0.5 + 0.875 = 1.375$.
3. The possible paths that link v_o to v_3 are only a path P_{e_1,e_2,e_3} of length three $v_o \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3$, then the minimum RD-length of this path is $d_{bRD}(v_o, v_3) =$
$$\frac{\mu(e_1)}{\sigma(v_0)\wedge\sigma(v_1)} + \frac{\mu(e_2)}{\sigma(v_1)\wedge\sigma(v_2)} + \frac{\mu(e_3)}{\sigma(v_2)\wedge\sigma(v_3)} = \frac{0.2}{0.4} + \frac{0.35}{0.4} + \frac{0.2}{0.3} = 0.5 + 0.875 + 0.666 = 2.041$$

Note that there is not path that links v_o to v_4, so by the function d_{bRD} as defined in Eq. (1), $d_{bRD}(v_o, v_4) = 0$.

The all possible paths that connect v_1 to other vertices in Fig. 3 are as follows:

1. The possible paths that link v_1 to v_o are only a path P_{e_2,e_3,e_4} of length three $v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3 \xrightarrow{e_4} v_o$, then the minimum RD-length of this path is

$d_{bRD}(v_1, v_o) =$
$$\frac{\mu(e_2)}{\sigma(v_1)\wedge\sigma(v_2)} + \frac{\mu(e_3)}{\sigma(v_2)\wedge\sigma(v_3)} + \frac{\mu(e_4)}{\sigma(v_3)\wedge\sigma(v_o)} = \frac{0.35}{0.4} + \frac{0.2}{0.3} + \frac{0.3}{0.3} = 0.875 + 0.666 + 1 = 2.541.$$

2. The possible paths that link v_1 to v_2 are only a path P_{e_2} of length one $v_1 \xrightarrow{e_2} v_2$, then the minimum RD-length of this path is

$$d_{bRD}(v_1, v_2) = \frac{\mu(e_2)}{\sigma(v_1)\wedge\sigma(v_2)} = \frac{0.35}{0.4\wedge 0.6} = \frac{0.35}{0.4} = 0.875.$$

3. The possible paths that link v_1 to v_3 are only a path P_{e_2,e_3} of length two $v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3$, then the minimum RD-length of this path is

$$d_{bRD}(v_1, v_3) = \frac{\mu(e_2)}{\sigma(v_1)\wedge\sigma(v_2)} + \frac{\mu(e_3)}{\sigma(v_2)\wedge\sigma(v_3)} = \frac{0.35}{0.4} + \frac{0.2}{0.3} = 0.875 + 0.666 = 1.541.$$

Note that there is not path that links v_1 to v_4, so by the function d_{bRD} as defined in Eq. (1), $d_{bRD}(v_1, v_4) = 0$.

Similarly, all b_{RD}-distances can be calculated between v_2 and the other vertices, as well as between v_3 and the other vertices in Fig. 3. At this point, we also need to demonstrate that all possible paths that link v_4 to the other vertices in Fig. 3 are as follows:

1. The various possible paths that link v_4 to v_o are two paths:

   i. There is a path P_{e_5} of length one $v_4 \xrightarrow{e_5} v_o$, then RD-length of this path is

$$L_{RD}(P_{e_5}) = \frac{\mu(e_5)}{\sigma(v_4)\wedge\sigma(v_o)} = \frac{0.2}{0.4\wedge 0.5} = \frac{0.2}{0.4} = 0.5.$$

   ii. There is a path P_{e_6,e_4} of length two $v_4 \xrightarrow{e_6} v_3 \xrightarrow{e_4} v_o$, then RD-length of this path is

$$L_{RD}(P_{e_6,e_4}) = \frac{\mu(e_6)}{\sigma(v_4)\wedge\sigma(v_3)} + \frac{\mu(e_4)}{\sigma(v_3)\wedge\sigma(v_o)} = \frac{0.25}{0.3} + \frac{0.3}{0.3} = 0.833 + 1 = 1.833.$$

   Therefore, evidently, the minimum RD-length of these two paths is $d_{bRD}(v_4, v_o) = 0.5$ and then, its path is P_{e_5}.

2. The diverse possible paths that link v_4 to v_1 are two paths:

   i. There is a path P_{e_5,e_1} of length two $v_4 \xrightarrow{e_5} v_o \xrightarrow{e_1} v_1$, then RD-length of this path is

$$L_{RD}(P_{e_5,e_1}) = \frac{\mu(e_5)}{\sigma(v_4)\wedge\sigma(v_o)} + \frac{\mu(e_1)}{\sigma(v_o)\wedge\sigma(v_1)} = \frac{0.2}{0.4} + \frac{0.2}{0.4} = 0.5 + 0.5 = 1$$

   ii.There is a path P_{e_6,e_4,e_1} of length three $v_4 \xrightarrow{e_6} v_3 \xrightarrow{e_4} v_o \xrightarrow{e_1} v_1$, then RD-length of this path is

$$L_{RD}(P_{e_6,e_4,e_1}) = \frac{\mu(e_6)}{\sigma(v_4)\wedge\sigma(v_3)} + \frac{\mu(e_4)}{\sigma(v_3)\wedge\sigma(v_o)} + \frac{\mu(e_1)}{\sigma(v_o)\wedge\sigma(v_1)} = \frac{0.25}{0.3} + \frac{0.3}{0.3} + \frac{0.2}{0.4} = 0.833 + 1 + 0.5 = 2.333.$$

3. The various possible paths that link v_4 to v_2 are two paths: i. There is a path P_{e_5,e_1,e_2} of length three $v_4 \xrightarrow{e_5} v_o \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2$, then RD-length of this path is

$$L_{RD}(P_{e_5,e_1,e_2}) = \frac{\mu(e_5)}{\sigma(v_4)\wedge\sigma(v_o)} + \frac{\mu(e_1)}{\sigma(v_o)\wedge\sigma(v_1)} + \frac{\mu(e_2)}{\sigma(v_1)\wedge\sigma(v_2)} = \frac{0.2}{0.4} + \frac{0.2}{0.4} + \frac{0.35}{0.4} = 0.5 + 0.5 + 0.875$$

$$= 1.875.$$

   ii.There is a path P_{e_6,e_4,e_1,e_2} of length four $v_4 \xrightarrow{e_6} v_3 \xrightarrow{e_4} v_o \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2$, then RD-length of this path is

$$L_{RD}(P_{e_6,e_4,e_1,e_2}) = frac\mu(e_6)\sigma(v_4)\wedge\sigma(v_3) + \frac{\mu(e_4)}{\sigma(v_3)\wedge\sigma(v_o)} + \frac{\mu(e_1)}{\sigma(v_o)\wedge\sigma(v_1)} + \frac{\mu(e_2)}{\sigma(v_1)\wedge\sigma(v_2)} = \frac{0.25}{0.3} + \frac{0.3}{0.3} + \frac{0.2}{0.4} + \frac{0.35}{0.4} = 0.833 + 1 + 0.5 + 0.875 = 3.208.$$

   Therefore, clearly, the minimum RD-length of these two paths is $d_{bRD}(v_4, v_2) = 1.875$ and then, its path is P_{e_5,e_1,e_2}.

Table 2 highlights all distances $d_{bRD}\left(v_i, v_j\right)$ representing the connections between vertex v_i and every other vertex v_j. For example, the distances $d_{bRD}\left(v_o, v_j\right)$ linking v_o to other vertices $v_j, \forall j = 1, 2, 3, 4$, are presented in the first row of Table 2, while the remaining distances $d_{bRD}\left(v_i, v_j\right)$ are listed in subsequent rows.

These values of distances $d_{bRD}\left(v_i, v_j\right)$ between any two different vertices v_i and v_j in the graph $RD_{\mathcal{FG}}(V, E)$, as summarized in Table 2, can readily provide immediate results about this graph, as outlined below.

Table 2. The b_{RD}-distances $\left(d_{bRD}\left(v_i, v_j\right)\right)$ between two vertices.

The vertices	V_0	V_1	V2	V3	V4
V_0	0	0.5	1.375	2.041	0
V_1	2.541	0	0.875	1.541	0
V2	1.666	2.166	0	0.666	0
V3	1	1.5	2.375	0	0
V4	0.5	1	1.875	0.833	0

- It is easily noticed that $d_{bRD}\left(v_i, v_j\right) \neq d_{bRD}\left(v_j, v_i\right), \forall v_i, v_j \in V = \{v_o, v_1, v_2, v_3, v_4\}$.
- It is clear that the axiom d_{b2} of Definition 2.4 is satisfied for all vertices $v_i, v_j \in V$. In fact, $d_{bRD}(v_i, v_i) = 0, \forall v_i \in V$ and $d_{bRD}(v_i, v_4) = 0$, for distinct vertices $v_i \neq v_4, \forall v_i \in \{v_o, v_1, v_2, v_3\}$
- It is accurately checked that the axiom d_{b5} of Definition 2.4 holds for all the values of distances $d_{bRD}\left(v_i, v_j\right)$. This means the axiom d_{b5} of the function d_{bRD} is fulfilled, i.e., $d_{bRD}\left(v_i, v_j\right) \leq M\left[d_{bRD}(v_i, v_k) + d_{bRD}\left(v_k, v_j\right)\right]$ for all $v_i, v_j, v_k \in V = \{v_o, v_1, v_2, v_3, v_4\}$ and where $M = 5.1$. For instance, if one takes the b_{RD}-distance that relates v_1 to v_o, then all achievable cases are as follows:
  1. $2.541 = d_{bRD}(v_1, v_o) \leq 5.1[d_{bRD}(v_1, v_2) + d_{bRD}(v_2, v_o)] = 5.1[0.875 + 1.666] = 12.959$.
  2. $2.541 = d_{bRD}(v_1, v_o) \leq 5.1[d_{bRD}(v_1, v_3) + d_{bRD}(v_3, v_o)] = 5.1[1.541 + 1] = 12.959$.
  3. $2.541 = d_{bRD}(v_1, v_o) \leq 5.1[d_{bRD}(v_1, v_4) + d_{bRD}(v_4, v_o)] = 5.1[0 + 0.5] = 2.55$.

     Therefore, the minimum possible value of M is 5.1 because it cannot be replaced by a smaller number that would satisfy all cases in inequality (2) in Definition 3.5.
- By Theorem 3.4, the reducible fuzzy digraph in Fig. 3 is a quasi-pseudo- b-metric. So, this digraph is called a quasi-pseudo- b-metric space of $RD_{\mathcal{FG}}(V, E)$.

Thus, in this example, we get $\mathcal{D} = 3M = 3(5.1) = 15.3$ which is the low-distortion $\mathcal{D}$-embedding of finite quasi-pseudo-(5.1)-metric space of a reducible fuzzy graph into space ℓ_∞. Here, note that $|V| = 5$ and $|V(R)| = 1$, since $V(R) = \{v_4\}$ which is the only one vertex that cannot be reached by a path ending by v_4 in Fig. 3. Then, by Theorem 3.7, there is a 15.3-embedding f of V into normed space $\ell_\infty^{k=4}$ due to $k = |V| - |V(R)| = 5 - 1 = 4$, where the vertex v_4 is removed during dimensionality reduction of this embedding.

To specify a 15.3 -embedding $f : (V, d_{bRD}) \to \ell_\infty^4$, we define 4 mappings $f_{v_o}, f_{v_1}, f_{v_2}, f_{v_3} : V \to \mathbb{R}$, the coordinates of the embedded vertices $\{v_o, v_1, v_2, v_3\}$. Here, the coordinates in ℓ_∞^4 are indexed by the vertices of V except the vertices in $V(R)$, i.e., $V/\{v_4\} = \{v_o, v_1, v_2, v_3\}$. Then, one define each f_{v_i} as the distance to an appropriate subset $A_{v_i} \subseteq V/\{v_4\}$, that is, the v_i th coordinate in ℓ_∞^4 is given by $f_{v_i}(v) = d_{bRD}(v, A_{v_i}) = \min_{u \in A_{v_i}} \{d_{bRD}(v, u)\}$ for all $i = 0, 1, 2, 3$ and all $v, u \in V/\{v_4\}$, where the subset $A_{v_i} = \{u \in V/\{v_4\}, d_{bRD}(v, u) \neq 0\}$ for all pairs $\{v, u\}$. Hence, we have a mapping f_{v_o} as follows:

$$f_{v_o}(v_o) = d_{bRD}(v_o, A_{v_o}) = \min_{u \in A_{v_o}} \{d_{bRD}(v_o, u)\} =$$
$$\min_{u \in A_{v_o}} \{d_{bRD}(v_o, v_1), d_{bRD}(v_o, v_2), d_{bRD}(v_o, v_3)\} =$$
$$\min_{u \in A_{v_o}} \{0.5, 1.375, 2.041\} = 0.5$$

, where $A_{v_o} = \{u \in V/\{v_4\}, d_{bRD}(v_o, u) \neq 0\} = \{v_1, v_2, v_3\}$

$$f_{v_o}(v_1) = d_{bRD}(v_1, A_{v_o}) = \min_{u \in A_{v_o}} \{d_{bRD}(v_1, u)\} =$$
$$\min_{u \in A_{v_o}} \{d_{bRD}(v_1, v_o), d_{bRD}(v_1, v_2), d_{bRD}(v_1, v_3)\}$$
$$= \min_{u \in A_{v_o}} \{2.541, 0.875, 1.541\} = 0.875$$

, where $A_{v_o} = \{u \in V/\{v_4\}, d_{bRD}(v, u) \neq 0\} = \{v_o, v_2, v_3\}$.

$$f_{v_o}(v_2) = d_{bRD}(v_2, A_{v_o}) = \min_{u \in A_{v_o}} \{d_{bRD}(v_2, u)\} =$$
$$\min_{u \in A_{v_o}} \{d_{bRD}(v_2, v_o), d_{bRD}(v_2, v_1), d_{bRD}(v_2, v_3)\} =$$
$$\min_{u \in A_{v_o}} \{1.666, 2.166, 0.666\} = 0.666$$

, where $A_{v_o} = \{u \in V/\{v_4\}, d_{bRD}(v, u) \neq 0\} = \{v_o, v_1, v_3\}$.

$$f_{v_o}(v_3) = d_{bRD}(v_3, A_{v_o}) = \min_{u \in A_{v_o}} \{d_{bRD}(v_3, u)\} =$$
$$\min_{u \in A_{v_o}} \{d_{bRD}(v_3, v_o), d_{bRD}(v_3, v_1), d_{bRD}(v_3, v_2)\} =$$
$$\min_{u \in A_{v_o}} \{1, 1.5, 2.375\} = 1.$$

, where $A_{v_o} = \{u \in V/\{v_4\}, d_{bRD}(v, u) \neq 0\} = \{v_o, v_1, v_2\}$.

In this example, as we have seen above, the values of functions f_{v_1}, f_{v_2} and f_{v_3} are also obtained in the same way as follows: $f_{v_i}(v_o) = 0.5, f_{v_i}(v_1) = 0.875, f_{v_i}(v_2) = 0.666, f_{v_i}(v_3) = 1$; for all $i = 1, 2, 3$.

By uncomplicated calculation, it can be checked that the coordinates f_i verify the inequalities (2) and (3), which means $\frac{1}{3(5.1)} d_{bRD}(v,u) \leq |f_i(v) - f_i(u)| \leq d_{bRD}(v,u)$ for all $i = 0,1,2,3$ and all $v, u \in V / \{v_4\}$.

Thus, it is interesting to show that a quasi-pseudo-5.1-metric space of the digraph in Fig. 3 can be embedded in a normed space $\ell_\infty^4 = (\mathbb{R}^4, \|x\|_\infty)$, where $x = (f_{v_o}, f_{v_1}, f_{v_2}, f_{v_3})$ and a norm for each x can be computed as follows: $\|x\|_\infty = \max_{i=0,1,2,3} \{|f_{v_i}|\} = \max\{|f_{v_0}|, |f_{v_1}|, |f_{v_2}|, |f_{v_3}|\} = 1$ with 15.3-embedding f.

The numerical example effectively reinforces the theoretical findings, providing a clear and practical demonstration of their validity and applicability. Its alignment with the theoretical framework enhances comprehension and confirms the significance of the results.

3.3.2. Real-World Application

In telecommunication networks, thousands or millions of nodes (like routers or servers) and edges (like communication links as fiber optic cables, wireless links with fuzzy weights) can be represented as a fuzzy digraph. Due to uncertainties like link failures, delays, or congestion, the distances between nodes may not follow the standard properties of a metric space. Instead, they form a quasi-pseudo-b-metric space, where distances may violate symmetry or the strict triangle inequality.

Computing b_{RD}-Distance : The b_{RD}-distance as defined in Fig. 1 offers an extensive framework for modeling multi-attribute relationships in the network. This distance captures the following key components:

- $\sigma(v_k)$: Denotes the status, strength, or significance of a node v_k (e.g., server capacity or node importance).
- $\mu(e_k)$: Represents the weight, reliability, or fuzzy membership of an edge e_k (e.g., the communication quality of a link).
- $L_{RD}(P)$: Refers to the path distance that evaluates all possible paths and selects the one with the minimal effective cost.

For each pair of nodes $\left(v_i, v_j\right)$, compute the b_{RD}-distance $d_{bRD}\left(v_i, v_j\right)$ based on all possible paths P between them. By modeling the network as a quasi-pseudo-b-metric space using the b_{RD}-distance, it becomes possible to:

- Quantify multi-attribute relationships (e.g., reliability, capacity, cost) between nodes.
- This approach simplifies the network by focusing on relevant attributes (e.g., node reliability, edge costs), unlike traditional distance metrics that enforce strict metric properties (like symmetry and the triangle inequality).

Reduce Dimensionality:

- Use a dimensionality reduction process to identify and remove redundant nodes $V(R)$, as defined in Definition 3.6, that do not significantly affect routing decisions in the network. Redundancy may arise from nodes with multiple alternative routes, minimal data flow, or nodes with marginal influence on the network's reliability or efficiency.
- Embed the reduced network into a low-dimensional normed space $\ell_\infty^k = \left(\mathbb{R}^k, \|x\|_\infty\right)$, where $k = |V| - |V(R)|$. Node positions in the k-low-dimensional space reflect its relationships with other nodes in terms of reliability, cost, and delay.

Embedding into ℓ_∞^k :

- First, less critical nodes are removed to simplify the network, focusing only on the most influential nodes.
- Each node $v_1, v_2, \ldots, v_n$ is embedded as a point in ℓ_∞^k, where the distances in the b-metric space are preserved in the normed space.
- The ℓ_∞^k norm is particularly useful for modeling the maximum influence or dominance, such as the worst-case delay, maximum path cost, or most critical link.
- The ℓ_∞^k norm ensures the embedding reflects the maximum deviation in attributes, making it ideal for worst-case modeling.
- The reduced dimensionality and embedding in ℓ_∞^k space speed up the pathfinding process, enabling faster, particularly in extensive telecommunication networks.

The b_{RD}-metric quantifies multi-attribute relationships (e.g., reliability, capacity, cost) between nodes, offering a flexible alternative to traditional distance metrics with rigid mathematical properties. Additionally, a dimensionality reduction process simplifies complex networks by removing inessential nodes while preserving critical structural and functional properties, thereby improving computational efficiency. Embedding the reduced network into low-dimensional normed spaces (e.g., ℓ_∞^k) enables faster pathfinding and real-time optimization. The fuzzy-based procedures effectively handle vagueness and ambiguity in real-world data, making them a strategic tool for decision-making in uncertain settings.

3.3.3. Comparison with Other distance Techniques

The b_{RD}-distance, as presented in Eq. (1), is introduced to address the limitations of traditional techniques, such as the shortest path, which fail to capture the intricacy and diversity of graph structures. In fuzzy graphs, a common traditional technique is the shortest path distance [15], which is based on the fuzzy interpretation of edge weights and is calculated as the minimum μ-length, where $L_\mu(P) = \sum_{k=1}^{n} \frac{1}{\mu((v_{k-1}, v_k))}$. Traditional metrics [35–37], including those based on the shortest path, rely solely on fixed edge weights and assume that the underlying graph of the fuzzy digraph is strongly connected. This assumption fails to account for essential contextual attributes such as the possibility of disconnections, as well as reliability, capacity, cost, and path quality.

As opposed to traditional techniques, the b_{RD}-distance overcomes these shortcomings by incorporating multiple edge-specific and path-specific attributes through custom transformations, including $\sigma(v)$ for vertex properties, $\mu(e)$ for edge weights, and additional parameters $L_{RD}(P)$ for path-level characteristics. This ensures a more detailed depiction of relationships between nodes.

Moreover, the b_{RD}-distance facilitates dimensionality reduction by recognizing and removing redundant nodes, permitting the graph to be embedded into a low-dimensional normed space (e.g., ℓ_∞^k) while preserving multi-attribute relationships. This feature is missing in traditional methods, making the b_{RD}-distance a more versatile and powerful approach for graph analysis.

The b_{RD}-distance not only overcomes the constraints of conventional methods but also offers a more adaptable and thorough framework for examining intricate graph structures, making it especially ideal for practical applications where reliability, capacity, and path quality are essential.

3.3.4. Future Directions and Applications of the b-Metric Approach

The b-metric-based approach generalizes traditional metric concepts, enabling more effective methods for graph embeddings and anomaly detection. It extends to massive datasets like social, biological, and sensor networks, handling non-Euclidean and high-dimensional data often found in data science. This approach supports dimensionality reduction, network optimization, and pathfinding, opening new research avenues. Mathematicians aim to provide rigorous proofs and theoretical foundations to support interdisciplinary applications in computer science, data science, and operations research.

4. Conclusions

The $\mathcal{D}$-embeddings offer a valuable and functional method for illustrating and representing the structures of reducible fuzzy digraphs with their b_{RD}-metric structures. They have gained significance in the industrial sector in recent years due to their effectiveness in handling large and noisy datasets across various application domains, as evidenced by the findings presented in this study. We first define the b_{RD}-distance function between any two different vertices in a reducible fuzzy graph $RD_{\mathcal{FG}}(V, E)$ and demonstrate that the b_{RD}-distance function is a quasi-pseudo-b-metric (see Theorem 3.4). This leads to the design of a new structure of a reducible fuzzy graph $RD_{\mathcal{FG}}(V, E)$, which is quasi-pseudo- b-metric space of $RD_{\mathcal{FG}}(V, E)$. Second, we provide a method for producing a low-distortion $\mathcal{D}$-embedding of the b_{RD}-metric of $RD_{\mathcal{FG}}(V, E)$ into the specific normed space ℓ_∞ (see Theorem 3.7) which incorporates ideas from fuzzy graph theory and functional analysis theory. These findings can ultimately be implemented to real-world scenarios, as demonstrated by a numerical example that validates and compares the proposed technique's ability to embed fuzzy graphs into normed spaces.

As future work, categorizing a reducible fuzzy graph as a quasi-pseudo- b_{RD}-metric structure contributes to the development of our findings, which will be extended and unified with various properties such as convergence and completeness within these structures. An additional path for future investigation is to explore how embeddings can be utilized to solve practical problems involving fuzzy data, such as in data mining, network analysis, decision making, and related fields.

Acknowledgements

The author would like to express thanks to Mustansiriyah University (www.uomustansiriyah.edu.iq), Baghdad, Iraq, for its support in the present work. The author extends genuine gratitude to the reviewers for their perceptive feedback and valuable suggestions.

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