Partial Fuzzy Soft Metric Space

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Received: Jul. 02, 2024; Accepted: Jan. 12, 2025

Matthew introduced a partial metric, sometimes known as a non-zero self-distance. This paper seeks to extend Matthew's notion to the fuzzy soft universe and create a partial fuzzy soft metric. Then we investigate the convergence of fuzzy soft sequences, fuzzy soft closed ball, fuzzy soft open ball, fuzzy soft map, and fuzzy soft fixed point theorems in a partial fuzzy soft metric space, as well as some fundamental characteristics.

Keywords: $\mathcal{F}_{\text{so}}$ - real number; partial $\mathcal{F}_{\text{so}}$ -metric space; $\mathcal{F}_{\text{so}}$ -mapping; $\mathcal{F}_{\text{so}}$ -Fixed point theorems

http://dx.doi.org/10.6180/jase.202602_29(2).0015

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1. Introduction

The idea of partial metric space was first introduced by Matthews in 1994 (see [1]). Fuzzy set theory, first presented by Zadeh [2], and the theory of soft sets, first proposed by Molodtsov [3] , are two kinds of mathematical tools for dealing with uncertainties. These techniques are useful for solving problems in all fields. Beaula and Gunaseeli [4], Sonam and Narayan [5], Kider [6], Sabri [7], Mohsen and Thiyab [8], and Mohsen [9] introduced the concept of fuzzy soft metric space. The concepts of fractional and fuzzy can also be combined to determine the optimal fuzzy control according to differential equations of fractional orders [10–13], as fractional dynamic systems have many engineering and physical applications such as electrical circuits, mechanics, control, electromagnetic processes, memory, genetic properties, and others. In this article, we defined fuzzy soft partial metric space in fuzzy soft points and $\mathcal{F}_{\text{so}}$ -sequence, $\mathcal{F}_{\text{so}}$ -Cauchy sequence, $\mathcal{F}_{\text{so}} -$ closed ball, $\mathcal{F}_{\text{so}} -$ open ball in the $\mathcal{F}_{\text{so}} -$ PMS and defined $\mathcal{F}_{\text{so}} -$ fixed point, $\mathcal{F}_{\text{so}} -$ continuous map in fuzzy soft mapping. Also, we have provided lemma explains the relationship between the $\mathcal{F}_{\text{so}} -$ a PMS $\left(\mathcal{F}_{\text{so}}(\mathcal{K}), \tilde{\tilde{\mathcal{P}}}\right)$ and $\mathcal{F}_{\text{so}} -$-metric space $\left(\mathcal{F}_{\text{so}}(\mathcal{K}), \tilde{\tilde{D}}\right)$ and we study fuzzy soft fixed point theorems in a partial fuzzy soft metric space.

2. Preliminaries

Throughout this article, $\mathcal{K}$ denotes an initial universe, E is the set of all parameters, $\mathrm{I} = [0,1]$, (Q, W) denotes a fuzzy soft set, PMS denotes partial metric space , and $\left\langle {k^n}_{\mu_{\mathrm{Q}}(\mathrm{e}_n)} \right\rangle$ denotes $\mathcal{F}_{s0} -$ sequence of element in this set. In this part, we provide some preliminary details on fuzzy soft sets.

Definition 2.1 . [2] The fuzzy set $\widetilde{\mathrm{X}}$ in $\mathcal{K}$ is defined by the map $\mu_{\widetilde{\mathrm{X}}} : \mathcal{K} \rightarrow \mathrm{I}$, where $\widetilde{\mathrm{X}}$ defined by $\widetilde{\mathrm{X}} = \left\{ \left(\mathrm{k}, \mu_{\widetilde{\mathrm{X}}}(\mathrm{k})\right) : \mathrm{k} \in \mathcal{K}, \mu_{\tilde{\mathrm{x}}}(\mathrm{k}) \in \mathrm{I} \right\}$, where $\mu_{\tilde{\mathrm{x}}}(\mathrm{k})$ is termed degree membership of k in $\widetilde{\mathrm{X}}$.

Definition 2.2 . [3] If W subset of E . Then the pair (Q, W) is soft set on $\mathcal{K}$ and defined by $Q_W = \{(\mathrm{e}, \mathrm{Q}_W(\mathrm{e})) : \mathrm{e} \in \mathrm{E}, \mathrm{Q}_W(\mathrm{e}) \in \mathrm{P}(\mathcal{K})\}$, such that Q a map defined by $\mathrm{Q} : W \rightarrow \mathrm{P}(\mathcal{K})$ and $\mathrm{P}(\mathcal{K})$ is a power set of $\mathcal{K}$.

Definition 2.3. [14] A fuzzy soft set on $\mathcal{K}$ is referred to as a set (Q, W), when Q is map $\mathrm{Q} : W \rightarrow \mathrm{I}^{\mathcal{K}}$, and $\left\{ \mathrm{Q}(\mathrm{e}) \in \mathrm{I}^{\mathcal{K}} : \mathrm{e} \in W \right\}$, $\mathcal{F}_{s0}(\mathcal{K})$ represents to the whole fuzzy soft collection, and $\mathcal{F}_{\text{so}}$ (short for fuzzy soft).

Remark 2.4. [14] The complement of a $\mathcal{F}_{50^-}$ (Q, W) is defined by $(Q, W)^C = \left(Q^C, W\right)$, where $Q^C : E \rightarrow I^{\mathcal{K}}$ is a mapping Given by $\mu Q^C(e) = 1 - \mu Q(e)$ for all $e \in W \subseteq E$., Note : W is called support of Q_W, and We get $\emptyset \neq \mathrm{Q}_W(\mathrm{e})$ for any $\mathrm{e} \in W$ and $\mathrm{Q}_W(\mathrm{e})$ equal to $\emptyset$ for all $\mathrm{e} \notin W$.

Definition 2.5. [15] A $\mathcal{F}_{\text{so}}{}^{-}(\mathrm{Q}, W)$ on $\mathcal{K}$ called.

1. An absolute $\mathcal{F}_{\mathrm{so}}$ -, denoted by C_W, if $\mu\mathrm{Q}(\mathrm{e}) = 1$ holds for every e in W,
2. A null $\mathcal{F}_{\mathrm{so}}$, denoted by $\emptyset$, if $\mu\mathrm{Q}(\mathrm{e}) = 0$ holds for every e in W,

You will now define the equality operation and the subset in $\mathcal{F}_{\mathrm{so}}$-set.,

Definition 2.6. [14] Let (Q_1, W) and $(\mathrm{Q}_2,\ \mathrm{B})$ be two $\mathcal{F}_{\mathrm{so}}$ - on $\mathcal{K}$.

1. Then (Q_1, W) is called a $\mathcal{F}_{\mathrm{so}}$ - subset of $(\mathrm{Q}_2,\ \mathrm{B})$ if $W \subseteq\ \mathrm{B}$, and $\mathrm{Q}_1(\mathrm{e}) \subseteq \mathrm{Q}_2(\mathrm{e})$ that is $\mu\mathrm{Q}_1(\mathrm{e}) \leq \mu\mathrm{Q}_2(\mathrm{e})$ for any e in W. We write $(\mathrm{Q}_1, W) \subseteq (\mathrm{Q}_2,\ \mathrm{B})$.,
2. The two $\mathcal{F}_{\mathrm{so}} - (\mathrm{Q}_1, W)$ and $(\mathrm{Q}_2,\ \mathrm{B})$ are called equal $\mathcal{F}_{\mathrm{so}}{}^{-}$, with symbolized by $(\mathrm{Q}_2,\ \mathrm{B}) = (\mathrm{Q}_1, W)$, if $(\mathrm{Q}_1, W) \subseteq (\mathrm{Q}_2,\ \mathrm{B})$ and $(\mathrm{Q}_2,\ \mathrm{B}) \subseteq (\mathrm{Q}_1, W)$.

Next, it's crucial to remember the **Definition** of the $\mathcal{F}_{\mathrm{so}}$ -point, which is represented by the symbol $\mathrm{k}_{\mu_{\mathrm{Q(e)}}}$.

Definition 2.7. [16] The $\mathcal{F}_{\mathrm{so}}$ - (Q, W) on $\mathcal{K}$ is namely $\mathcal{F}_{\mathrm{so}}$ - point and denoted by $\mathrm{k}\text{-}_{\mathrm{Q}(e)}$, if $\mathrm{k} \in \mathcal{K}$ and for every e in W,

$$\mu_{\mathrm{Q(e)}} = \begin{cases} \sigma \text{ if } k = \mathrm{k}_0 \in \mathcal{K} \text{ and } \mathrm{e} = \mathrm{e}_0 \in W, \\ 0 \text{ if } \mathrm{k} \in \mathcal{K} - \{\mathrm{k}_0\} \text{ ore } \in W - \{\mathrm{e}_0\}, \text{where } \sigma \in (0,1] \end{cases}$$

Definition 2.8. [17] Assume $\mathrm{k}_{\mu^1_{\mathrm{Q}(e_1)}}$ and $\mathrm{k}_{\mu_{\mathrm{Q}(e_2)}}$ are two $\mathcal{F}_{\mathrm{so}^-}$-point on $\mathcal{K}$ and (Q_1, W) a $\mathcal{F}_{\mathrm{so}}{}^{-}$ then:

1. The $\mathcal{F}_{\mathrm{so}}{}^{-}$point $\mathrm{k}\text{-}_{\mathrm{Q(e)}}$ is called belongs to (Q_1, W) if for every e in W, We get $\mathrm{Q}(\mathrm{e}) \subseteq \mathrm{Q}_1(\mathrm{e})$. and symbolized by $\mathrm{k}\text{-}_{\mathrm{Q}_{\mathrm{Q(e)}}} \in (\mathrm{Q}_1, W)$,
2. Both of the $\mathcal{F}$ so ${}^{-}$points $\mathrm{k}^1{}_{\mathrm{Q}_{\mathrm{Q}(\mathrm{e}_1)}}$ and $\mathrm{k}^2{}_{\mu_{\mathrm{Q}(\mathrm{e}_2)}}$ on $\mathcal{K}$ are namely equal $\mathcal{F}_{\mathrm{so}^-}$ if $\mathrm{k}^1 = \mathrm{k}^2, \mathrm{e}_1 = \mathrm{e}_2$ and $\mu_{\mathrm{Q}(\mathrm{e}_1)} = \mu_{\mathrm{Q}(\mathrm{e}_2)}$.
3. The $\mathcal{F}_{\mathrm{so}}$ -point $\mathrm{k}\text{-}_{\mathrm{Q^c(e)}}$ is called complement $\mathcal{F}_{\mathrm{so}'-\text{ point}}$ of $\mathcal{F}_{\mathrm{so}}$ - point $\mathrm{k}_{\mu_{\mathrm{Q(e)}}}$ if for every e in W and $\mathrm{k} \in \mathcal{K}$, $\mu_{Qc(e)} = \begin{cases} 1 - \mu_{Q(e)}, \text{ ifk } = \mathrm{k}_0 \in \mathcal{K} \text{ and } \mathrm{e} = \mathrm{e}_0 \in W \\ 0, \text{ifk} \in \mathcal{K} - \{\mathrm{k}_0\} \text{ ore } \in W - \{\mathrm{e}_0\}. \end{cases}$

Definition 2.9. [18] Consider the $\mathcal{F}_{s0^-}$ $(\mathrm{Q}_2,\ \mathrm{B})$ and (Q_1, W) on $\mathcal{K}$. Then:

1. $(Q_1, W) \cup (Q_2, B) = (Q_3, C)$, where $B \cup W$ equal C and for any $e \in C, k \in \mathcal{K}$, $\mu Q_3(\mathrm{e})(\mathrm{k}) = \begin{cases} \mu Q_1(\mathrm{e})(k), \text{ if } \mathrm{e} \in W - B, k \in \mathcal{K} \\ \mu Q_2(\mathrm{e})(k), \text{ if } \mathrm{e} \in B - W, k \in \mathcal{K} \\ \max\left[\mu Q_1(\mathrm{e})(k), \mu Q_2(\mathrm{e})(k)\right], \text{ if } \mathrm{e} \in W \cap B, k \in \mathcal{K} \end{cases}$
2. $(Q_1, W) \cap (Q_2, B) = (Q_3, C)$, where C equal $W \cup B$ and for every e in C, $\mu Q_3(\mathrm{e})(\mathrm{k}) = \begin{cases} \mu Q_1(\mathrm{e})(k), \text{ if } \mathrm{e} \in W - B, k \in \mathcal{K} \\ \mu Q_2(\mathrm{e})(k), \text{ if } \mathrm{e} \in B - W, k \in \mathcal{K} \\ \min\left[\mu Q_1(\mathrm{e})(k), \mu Q_2(\mathrm{e})(k)\right], \text{ if } \mathrm{e} \in W \cap B, k \in \mathcal{K}. \end{cases}$

Definition 2.10. [19] If R real integers and W subset of E and the set of all non-empty bounded fuzzy subsets of R is denoted as $\mathcal{F}^{\mathrm{B(R)}}$, then (R, W) namely $\mathcal{F}_{\mathrm{so}}$ - real set on R and is defined as a set of $\mathrm{R}_W = \left\{(\mathrm{e}, \mathrm{R}_W(\mathrm{e})) : \text{for every e in } W, \mathrm{R}_W(\mathrm{e}) \in \mathcal{F}^{\mathrm{B(R)}}\right\}$, where $\mathrm{R} : W \to \mathcal{F}^{\mathrm{B(R)}}$.$W$ is called support of R_W.

Definition 2.11. [19] Assume (R, W) is a $\mathcal{F}_{\mathrm{so}}$ - real number in R, represented by (r, W) (shortly $\bar{\mathrm{r}}$), when is a singleton $\mathcal{F}_{\mathrm{so}^-}$ real set, such as $R(W)$ signified the set of all $\mathcal{F}_{\mathrm{so}}$ - real values and $R^+(W)$ denoted a set of all $\mathcal{F}_{\mathrm{so}}$ - real a positive values.

Definition 2.12. [4] Let $\mathcal{F}_{\mathrm{so}}\ (\mathcal{K})$ be the family of all a non-empty $\mathcal{F}_{\mathrm{so}} -$ on $\mathcal{K}$, a $\mathcal{F}_{\mathrm{so}} -$ metric space is denoted by $\left(\mathcal{F}_{\mathrm{so}}\ (\mathcal{K}), \tilde{\tilde{\mathrm{d}}}\right)$, when $\tilde{\tilde{\mathrm{d}}} : \mathcal{F}_{\mathrm{so}}\ (\mathcal{K}) \times \mathcal{F}_{\mathrm{so}}\ (\mathcal{K}) \to \mathrm{R}^+(W)$ be a map that fulfills the following conditions for any $\mathrm{k}^1{}_{\mu_{\mathrm{Q}(\mathrm{e}_1)}}, \mathrm{k}^2{}_{\mu_{\mathrm{Q}(\mathrm{e}_2)}}, \mathrm{k}_{\mu_{\mathrm{Q}(\mathrm{e}_3)}} \in \mathcal{F}_{\mathrm{so}}(\mathcal{K})$.

1. $0 \leq \tilde{\tilde{d}}\left(\mathrm{k}^1{}_{\mu_{\mathrm{Q}(\mathrm{e}_1)}}, \mathrm{k}^2{}_{\mu_{\mathrm{Q}(\mathrm{e}_2)}}\right)$.
2. $k^1{}_{\mu_{\mathrm{Q}(\mathrm{e}_1)}} = k^2{}_{\mu_{\mathrm{Q}(\mathrm{e}_2)}}$ if and only if $\tilde{\tilde{d}}\left(\mathrm{k}^1{}_{\mu_{\mathrm{Q}(\mathrm{e}_1)}}, \mathrm{k}^2{}_{\mu_{\mathrm{Q}(\mathrm{e}_2)}}\right) = 0$.
3. $\tilde{\mathrm{d}}\left(\mathrm{k}^1{}_{\mu_{\mathrm{Q}(\mathrm{e}_1)}}, \mathrm{k}^2{}_{\mu_{\mathrm{Q}(\mathrm{e}_2)}}\right) = \tilde{\mathrm{d}}\left(\mathrm{k}^2{}_{\mu_{\mathrm{Q}(\mathrm{e}_2)}}, \mathrm{k}^1{}_{\mu_{\mathrm{Q}(\mathrm{e}_1)}}\right)$.
4. $\tilde{\mathrm{d}}\left(\mathrm{k}^1{}_{\mu_{Q(e_1)}}, \mathrm{k}^2{}_{\mu_{\mathrm{Q}(e_2)}}\right) + \tilde{\mathrm{d}}\left(\mathrm{k}^2{}_{\mu_{Q(e_2)}}, \mathrm{k}^3{}_{\mu_{Q(e_3)}}\right) \geq \tilde{\mathrm{d}}\left(\mathrm{k}^1{}_{\mu_{Q(e_1)}}, \mathrm{k}^3{}_{\mu_{\mathrm{Q}(e_3)}}\right)$

3. Partial fuzzy soft metric space

In this section, we define the partial fuzzy soft metric and study the convergence of a sequence of fuzzy soft points in this space to its fundamental features.

A fuzzy soft Partial metric on a nonempty fuzzy soft set $\mathcal{F}_{\mathrm{so}}\ (\mathcal{K})$ is a function $\tilde{\tilde{\tilde{\mathcal{P}}}} : \mathcal{F}_{\mathrm{so}}(\mathcal{K}) \times \mathcal{F}_{\mathrm{so}}(\mathcal{K}) \to \mathrm{R}^+(W)$ such that satisfy the following statements for all $\mathrm{k}^1\mu_{\mu_{\mathrm{Q}(\mathrm{e}_1)}}, \mathrm{k}^2{}_{\mu_{\mathrm{Q}(\mathrm{e}_2)}}, \mathrm{k}^3{}_{\mu_{\mathrm{Q}(\mathrm{e}_3)}} \in \mathcal{F}_{\mathrm{so}}(\mathcal{K})$.

1. $\tilde{\tilde{\tilde{\mathcal{P}}}}\left(\mathrm{k}^1{}_{\mu_{\mathrm{Q}(\mathrm{e}_1)}},, \mathrm{k}^2{}_{\mu_{\mathrm{Q}(\mathrm{e}_2)}}\right) \geq \tilde{\tilde{\tilde{\mathcal{P}}}}\left(\mathrm{k}^1{}_{\mu_{\mathrm{Q}(\mathrm{e}_1)}}, \mathrm{k}^1{}_{\mu_{\mathrm{Q}(\mathrm{e}_1)}}\right)$
2. $\tilde{\mathcal{P}}\left(\mathrm{k}^1{}_{\mu_Q(\mathrm{e}_1)}, \mathrm{k}^2{}_{\mu_Q(\mathrm{e}_2)}\right) = \tilde{\mathcal{P}}\left(\mathrm{k}^1{}_{\mu_Q(\mathrm{e}_1)}, \mathrm{k}^1{}_{\mu_Q(\mathrm{e}_1)}\right) = \tilde{\tilde{\tilde{\mathcal{P}}}}\left(\mathrm{k}^2{}_{\mu_Q(\mathrm{e}_2)}, \mathrm{k}^2{}_{\mu_Q(\mathrm{e}_2)}\right)$ if and only if $k^1{}_{\mu_Q(\mathrm{e}_1)} = k^2{}_{\mu_{Q(\mathrm{e}_2)}}$
3. $\widetilde{\mathcal{P}}\left(\mathrm{k}^1{}_{\mu_{\mathrm{Q}(\mathrm{e}_1)}},, \mathrm{k}^2{}_{\mu_{\mathrm{Q}(e_2)}}\right) = \tilde{\tilde{\tilde{\mathcal{P}}}}\left(\mathrm{k}^2{}_{\mu_{\mathrm{Q}(\mathrm{e}_2)}},, \mathrm{k}^1{}_{\mu_{\mathrm{Q}(\mathrm{e}_1)}}\right)$
4. $\widetilde{\mathcal{P}}\left(\mathrm{k}^1{}_{\mu_{\mathrm{Q}(\mathrm{e}_1)}},, \mathrm{k}^3{}_{\mu_{\mathrm{Q}(e_3)}}\right) \leq \widetilde{\mathcal{P}}\left(\mathrm{k}^1{}_{\mu_{\mathrm{Q}(\mathrm{e}_1)}},, \mathrm{k}^2{}_{\mu_{\mathrm{Q}(e_2)}}\right) + \widetilde{\mathcal{P}}\left(\mathrm{k}^2{}_{\mu_{\mathrm{Q}(\mathrm{e}_2)}},, \mathrm{k}^3{}_{\mu_{\mathrm{Q}(e_3)}}\right) - \widetilde{\mathcal{P}}\left(\mathrm{k}^2{}_{\mu_{\mathrm{Q}(\mathrm{e}_2)}},, \mathrm{k}^2{}_{\mu_{\mathrm{Q}(e_2)}}\right)$

If $\mathcal{F}_{\mathrm{so}}(\mathcal{K})$ is a nonempty $\mathcal{F}_{\mathrm{so}}-$ set of the universe $\mathcal{K}$ and $\tilde{\tilde{\mathcal{P}}}$ is a partial $\mathcal{F}_{\mathrm{so}}-$ metric on it, then the pair $(\mathcal{F}_{\mathrm{so}}(\mathcal{K}), \tilde{\mathcal{P}})$ is referred to as a partial $\mathcal{F}_{\mathrm{so}}-$ metric space. It is simple for showing that the $\mathcal{F}_{\mathrm{so}}-$ function $\tilde{\tilde{D}}: \mathcal{F}_{\mathrm{so}}(\mathcal{K}) \times \mathcal{F}_{\mathrm{so}}(\mathcal{K}) \to \mathrm{R}^{+}(W)$ defined as

$$\tilde{\tilde{D}}\left(k^{1}{}_{\mu_Q(e_1)}, k^{2}{}_{\mu_Q(e_2)}\right) = \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}_{\mu_Q(e_1)}, \mathrm{k}^{2}{}_{\mu_Q(e_2)}\right) - \tilde{\tilde{P}}\left(\mathrm{k}^{1}{}_{\mu_Q(e_1)}, \mathrm{k}^{1}{}_{\mu_Q(e_2)}\right) \tag{1.1}$$

Fulfills the rules for a $\mathcal{F}_{\mathrm{so}}-$ metric on $\mathcal{F}_{\mathrm{so}}(\mathcal{K})$, hence it is a (standard) $\mathcal{F}_{\mathrm{so}}$ - metric on $\mathcal{F}_{\mathrm{so}}(\mathcal{K})$. In a $\mathcal{F}_{\mathrm{so}}$ $-PMS$, the concepts of $\mathcal{F}_{\mathrm{so}}-$ convergence, $\mathcal{F}_{\mathrm{so}}-$ Cauchy sequence, completeness, and continuity are defined as follows.

Definition 3.1

1. A $\mathcal{F}_{\mathrm{so}}-$ sequence $\left\{k^{n}{}_{\mu_{Q(e_n)}}\right\}$ in the $\mathcal{F}_{\mathrm{so}}$ $-$ PMS $(\mathcal{F}_{\mathrm{so}}(\mathcal{K}), \tilde{\mathcal{P}})$ converges to $\mathrm{k}^{0}{}_{\mu_Q(e_0)}$ if and only if $\tilde{\mathcal{P}}\left(\mathrm{k}^{0}{}_{\mu_Q(e_0)}, \mathrm{k}^{0}{}_{\mu_Q(\mathrm{e}_0)}\right) = \lim_{n\to\infty} \tilde{\mathcal{P}}\left(\mathrm{k}^{0}{}_{\mu_{Q(e_0)}}, \mathrm{k}^{\mathrm{n}}{}_{\mu_Q(\mathrm{e}_\mathrm{n})}\right)$
2. A $\mathcal{F}_{\mathrm{so}}-$ sequence $\left\{k^{n}{}_{\mu_{Q(e_n)}}\right\}$ in the $\mathcal{F}_{\mathrm{so}}$ $-$ PMS $\left(\mathcal{F}_{\mathrm{so}}(\mathcal{K}), \tilde{\tilde{\mathcal{P}}}\right)$ is considered a $\mathcal{F}_{\mathrm{so}}$-Cauchy sequence if $\lim_{n,m\to\infty} \tilde{\mathcal{P}}\left(\mathrm{k}^{\mathrm{n}}{}_{\mu_Q(e_n)}, \mathrm{k}^{\mathrm{m}}{}_{\mu_{\ell(e_m)}}\right)$ exists and is finite.
3. A $\mathcal{F}_{50}-$ PMS $\left(\mathcal{F}_{50}(\mathcal{K}), \tilde{\tilde{\mathcal{P}}}\right)$ is considered complete when every $\mathcal{F}_{50}$ - Cauchy sequence $\left\{k^{n}{}_{\mu_Q(e_n)}\right\}$ in $\mathcal{F}_{s0}(\mathcal{K})$ converges to a $\mathcal{F}_{\mathrm{so}}$ - point $\mathrm{k}^{0}{}_{\mu_Q(e_0)} \in \mathcal{F}_{\mathrm{so}}(\mathcal{K})$ such that $\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{0}{}_{\mu_Q(e_0)}, \mathrm{k}^{0}{}_{\mu_Q(e_0)}\right) = \lim_{n,m\to\infty} \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{\mathrm{n}}{}_{\mu_Q(e_n)}, \mathrm{k}^{\mathrm{m}}{}_{\mu_Q(e_m)}\right)$

Definition 3.2 Assume $\left(\mathcal{F}_{\mathrm{so}}(\mathcal{K}), \tilde{\tilde{\mathcal{P}}}\right)$ be a $\mathcal{F}_{\mathrm{so}}$ $^{-}$a PMS and $\bar{\mathrm{r}} \in \mathrm{R}^{+}(W)$. An open ball centered at $\mathcal{F}_{\mathrm{so}}-$ point $\mathrm{k}_{\mu_{Q(e)}} \in \mathcal{F}_{\mathrm{so}}(\mathcal{K})$ and radius $\mathrm{r}^{-}$is a set of all $\mathcal{F}_{\mathrm{so}}$ - points $\mathrm{k}_{\mu_{Q(\mathrm{a})}}$ of $\mathcal{F}_{\mathrm{so}}(\mathcal{K})$ such that $\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}_{\mu_{Q(\mathrm{a})}}, \mathrm{k}_{\mu_{Q(\mathrm{e})}}\right) < \mathrm{r}^{-}$. It is symbolized by

$$\tilde{\tilde{\mathcal{P}}}_{\bar{\mathrm{r}}}\left(\mathrm{k}_{\mu_Q(\mathrm{e})}\right) \text{.i.e.} \tilde{\tilde{\mathcal{P}}}_{\bar{\mathrm{r}}}\left(\mathrm{k}_{\mu_Q(\mathrm{e})}\right) = \left\{\mathrm{k}_{\mu_{Q(a)}} \in \mathcal{F}_{SO}(\mathcal{K}) : \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}_{\mu_Q(a)}, \mathrm{k}_{\mu_Q(e)}\right) < \mathrm{r}^{-}\right\}$$

The$\mathcal{F}_{\mathrm{so}}$ $-$ closed ball symbolized by

$$\tilde{\tilde{\mathcal{P}}}_{\bar{\mathrm{r}}}\left[\mathrm{k}_{\mu_{Q(e)}}\right] = \left\{\mathrm{k}_{\mu_{Q(a)}} \in \mathcal{F}_{\mathrm{so}}(\mathcal{K}) : \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}_{\mu_{Q(a)}}, \mathrm{k}_{\mu_{Q(e)}}\right) \le \bar{\mathrm{r}}\right\}$$

.

Definition 3.3 Let $(\mathcal{F}_{\mathrm{so}}(\mathcal{K}_1), \tilde{\mathcal{P}}_1, \mathrm{E}_1)$ and $(\mathcal{F}_{\mathrm{so}}(\mathcal{K}_2), \tilde{\mathcal{P}}_2, \mathrm{E}_2)$ be two $\mathcal{F}_{\mathrm{so}}$ $-$ PMS, respectively. If $\mathrm{T}: \mathcal{K}_1 \to \mathcal{K}_2$ and $\psi: \mathrm{E}_1 \to \mathrm{E}_2$ are both maps, when E_2 and E_1 are the parameter sets for the crisp sets $\mathcal{K}_2$ and $\mathcal{K}_1$. Then the map $\mathrm{T}\psi = (\mathrm{T}, \psi): (\mathcal{F}_{\mathrm{so}}(\mathcal{K}_1), \tilde{\mathcal{P}}_1, \mathrm{E}_1) \to (\mathcal{F}_{\mathrm{so}}(\mathcal{K}_2), \tilde{\mathcal{P}}_2, \mathrm{E}_2)$ is called as a $\mathcal{F}_{\mathrm{so}}$ -map.

Definition 3.4 A $\mathcal{F}_{\mathrm{so}}$-mapping $\mathrm{T}\psi: \mathcal{F}_{\mathrm{so}}(\mathcal{K}) \to \mathcal{F}_{\mathrm{so}}(\mathcal{K})$ is said to be $\mathcal{F}_{\mathrm{so}}$-continuous at $\mathrm{k}^{0}{}_{\mu_{Q(e_0)}} \tilde{\in} \mathcal{F}_{\mathrm{so}}(\mathcal{K})$ if for any $\bar{e} > 0$, there exists $\bar{\delta} > 0$ such that $\mathrm{T}\psi\left(\tilde{\mathcal{P}}_{\bar{\delta}}\left(\mathrm{k}^{0}{}_{\mu_Q(\mathrm{e}_0)}\right)\right) \tilde{\subseteq} \tilde{\mathcal{P}}_{\bar{e}}\left(\mathrm{T}\psi\left(\mathrm{k}^{0}{}_{\mu_{Q(\mathrm{e}_0)}}\right)\right)$.

Definition 3.5 Let $\mathrm{T}\psi: \mathcal{F}_{\mathrm{so}}(\mathcal{K}) \to \mathcal{F}_{\mathrm{so}}(\mathcal{K})$ be a $\mathcal{F}_{\mathrm{so}}$ - map and $\mathrm{k}^{1}{}_{\mu_{Q(\mathrm{e}_1)} \in \mathcal{F}_{\mathrm{so}}(\mathcal{K}) \text{ be a}}$ $\mathcal{F}_{\mathrm{so}}$ - element. If $T\psi\left(\mathrm{k}^{1}{}_{\mu_{Q(e_1)}}\right) = \mathrm{k}^{1}{}_{\mu_{Q(e_1)}}$. Then $\mathrm{k}^{1}{}_{\mu_{Q(\mathrm{e}_1)}}$ is called a $\mathcal{F}_{\mathrm{so}}$ - fixed point of $\mathrm{T}\psi$.The following lemma explains the relationship between the $\mathcal{F}_{\mathrm{so}}$- a PMS $(\mathcal{F}_{\mathrm{so}}(\mathcal{K}), \tilde{\mathcal{P}})$ and $\mathcal{F}_{\mathrm{so}}$-metric space $\left(\mathcal{F}_{\mathrm{so}}(\mathcal{K}), \tilde{\tilde{D}}\right)$ in terms of the $\mathcal{F}_{\mathrm{so}}$ - Cauchy sequence.

Lemma 3.6.

1. a $\mathcal{F}_{\mathrm{so}}$ - sequence $\left\{k^{n}{}_{\mu_{Q(e_n)}}\right\}$ is a $\mathcal{F}_{\mathrm{so}}-$ Cauchy sequence in the $\mathcal{F}_{\mathrm{so}}$ - metric space $(\mathcal{F}_{\mathrm{so}}(\mathcal{K}), \tilde{\mathrm{D}})$ If and only if the $\mathcal{F}_{\mathrm{so}}$ -sequence $\left\{k^{n}{}_{\mu_{Q(e_n)}}\right\}$ is $\mathcal{F}_{\mathrm{so}}$ -Cauchy sequence in the $\mathcal{F}_{50}-$ PMS $\left(\mathcal{F}_{s0}(\mathcal{K}), \tilde{\tilde{P}}\right)$.
2. the $\mathcal{F}_{\mathrm{so}}$ - metric space $(\mathcal{F}_{\mathrm{so}}(\mathcal{K}), \tilde{\mathrm{D}})$ is complete If and only if a $\mathcal{F}_{\mathrm{so}}$ $-PMS$ $\left(\mathcal{F}_{\mathrm{so}}(\mathcal{K}), \tilde{\tilde{\mathcal{P}}}\right)$ is complete.. Additionally

$$\begin{gathered}\lim_{\mathrm{n}\to\infty} \tilde{\tilde{\mathcal{D}}}\left(\mathrm{k}^{0}{}_{\mu_{Q(e_0)}}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{Q(e_n)}}\right\} = \bar{0} \text{ if and only if } \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{0}{}_{\mu_{Q(e_0)}}, \mathrm{k}^{0}{}_{\mu_{Q(e_0)}}\right\} = \\ \lim_{\mathrm{n}\to\infty} \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{0}{}_{\mu_{Q(e_0)}}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{Q(e_n)}}\right\} = \lim_{n,m\to\infty} \lim_{\mathrm{n}\to\infty} \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{\mathrm{n}}{}_{\mu_{Q(e_n)}}, \mathrm{k}^{\mathrm{m}}{}_{\mu_{Q(e_m)}}\right\}\end{gathered} \tag{1.2}$$

Proof. (1) First, we prove that each $\mathcal{F}_{\mathrm{so}}$ -Cauchy sequence in $\left(\mathcal{F}_{\mathrm{so}}(\mathcal{K}), \tilde{\tilde{\mathcal{P}}}\right)$ is also a $\mathcal{F}_{\mathrm{so}}$ $^{-}$ Cauchy sequence in $(\mathcal{F}_{\mathrm{so}}(\mathcal{K}), \tilde{\mathrm{D}})$. Let $\left\{k^{n}{}_{\mu_{Q(e_n)}}\right\}$ be a $\mathcal{F}_{\mathrm{so}}$ - Cauchy sequence in $(\mathcal{F}_{\mathrm{so}}(\mathcal{K}), \tilde{\mathcal{P}})$. Then there exists $\delta \in R$ such that, for any $\varepsilon > 0$, there is $k \in \mathbb{N}$ for every n,m > k where $\left|\tilde{\tilde{P}}\left(\mathrm{k}^{\mathrm{n}}{}_{\mu_Q(\mathrm{e}_\mathrm{n})}, \mathrm{k}^{\mathrm{m}}{}_{\mu_Q(\mathrm{e}_\mathrm{m})}\right) - \delta\right| < \frac{\varepsilon}{2}$

Therefore by use (1.1) we get.

$$\begin{aligned}\tilde{\tilde{D}}\left(\mathrm{k}^n_{\mu_Q(e_n)},\mathrm{k}^m_{\mu_Q(e_m)}\right) &= \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^n_{\mu_Q(e_n)},\mathrm{k}^m_{\mu_Q(e_m)}\right)-\tilde{\mathcal{P}}\left(\mathrm{k}^n_{\mu_Q(e_n)},\mathrm{k}^n_{\mu_Q(e_n)}\right)\\ &= \left|\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^n_{\mu_Q(e_n)},\mathrm{k}^m_{\mu_Q(e_m)}\right)-\delta+\delta-\tilde{\mathcal{P}}\left(\mathrm{k}^n_{\mu_Q(e_n)},\mathrm{k}^n_{\mu_Q(e_n)}\right)\right|\\ &\leq \left|\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^n_{\mu_Q(e_n)},\mathrm{k}^m_{\mu_Q(e_m)}\right)-\delta\right|+\left|\delta-\tilde{\mathcal{P}}\left(\mathrm{k}^n_{\mu_Q(e_n)},\mathrm{k}^n_{\mu_Q(e_n)}\right)\right|\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

For any $n,m > k$. Likewise, we prove that $\tilde{\tilde{D}}\left(\mathrm{k^m}_{\mu_Q(e_m)},\mathrm{k^n}\text{-}_{\mathrm{Q(e_n)}}\right) < \varepsilon$ for all $n,m > k$ .We conclude that $\left\{k^n{}_{\mu_{Q(e_n)}}\right\}$ is a $\mathcal{F}_{\mathrm{so}}$ -Cauchy sequence in $\left(\mathcal{F}_{\mathrm{so}}\left(\mathcal{K}\right),\tilde{\tilde{D}}\right)$. We now show that each $\mathcal{F}_{\mathrm{so}}$ - Cauchy sequence $\left\{k^n{}_{\mu_{Q(e_n)}}\right\}$ in $\left(\mathcal{F}_{\mathrm{so}}\left(\mathcal{K}\right),\tilde{\tilde{\mathrm{D}}}\right)$ is also a $\mathcal{F}_{\mathrm{so}}$ - Cauchy sequence in $\left(\mathcal{F}_{\mathrm{so}}\left(\mathcal{K}\right),\tilde{\tilde{P}}\right)$. Assume $\varepsilon=\frac{1}{2}$. Then there exists $K_0 \in \mathbb{N}$, for any $n,m > K_0$,such that $\tilde{\tilde{D}}\left(\mathrm{k^m}_{\mu_{\mathrm{Q}}(e_{\mathrm{m}})},\mathrm{k^n}\,\mu_{\mathrm{Q}(e_{\mathrm{n}})}\right) < \frac{1}{2}$.

$$\begin{aligned}&\text{Since}\quad \tilde{\tilde{D}}\left(\mathrm{k}^n_{\mu_Q(e_n)},\mathrm{k}^0_{\mu_Q(e_0)}\right)+\tilde{\tilde{P}}\left(\mathrm{k}^n_{\mu_Q(e_n)},\mathrm{k}^n_{\mu_Q(e_n)}\right)=\tilde{\tilde{D}}\left(\mathrm{k}^0_{\mu_Q(e_0)},\mathrm{k}^n_{\mu_Q(e_n)}\right)+\tilde{\tilde{P}}\left(\mathrm{k}^0_{\mu_Q(e_0)},\mathrm{k}^0_{\mu_Q(e_0)}\right),\\ &\text{hence}\quad \left|\tilde{\tilde{P}}\left(\mathrm{k}^n_{\mu_Q(e_n)},\mathrm{k}^n_{\mu_Q(e_n)}\right)\right|=\left|\tilde{\tilde{D}}\left(\mathrm{k}^0_{\mu_Q(e_0)},\mathrm{k}^n_{\mu_Q(e_n)}\right)+\tilde{\tilde{P}}\left(\mathrm{k}^0_{\mu_Q(e_0)},\mathrm{k}^0_{\mu_Q(e_0)}\right)-\tilde{\tilde{D}}\left(\mathrm{k}^n_{\mu_Q(e_n)},\mathrm{k}^0_{\mu_Q(e_0)}\right)\right|,\\ &\qquad\leq \tilde{\tilde{D}}\left(\mathrm{k}^0_{\mu_Q(e_0)},\mathrm{k}^n_{\mu_Q(e_n)}\right)+\left|\tilde{\tilde{P}}\left(\mathrm{k}^0_{\mu_Q(e_0)},\mathrm{k}^0_{\mu_Q(e_0)}\right)\right|+\tilde{\tilde{D}}\left(\mathrm{k}^n_{\mu_Q(e_n)},\mathrm{k}^0_{\mu_Q(e_0)}\right),\\ &\qquad\leq 2\tilde{\tilde{D}}\left(\mathrm{k}^n_{\mu_Q(e_n)},\mathrm{k}^0_{\mu_Q(e_0)}\right)+\left|\tilde{\tilde{P}}\left(\mathrm{k}^0_{\mu_Q(e_0)},\mathrm{k}^0_{\mu_Q(e_0)}\right)\right|<\bar{1}+\left|\tilde{\tilde{P}}\left(\mathrm{k}^0_{\mu_Q(e_0)},\mathrm{k}^0_{\mu_Q(e_0)}\right)\right|.\end{aligned}$$

As a result, there is $\bar{a} \in \mathrm{R}(W)$ (fuzzy soft real number) such that a $\mathcal{F}_{\mathrm{so}}$ -subsequence $\left\{\tilde{\tilde{P}}\left(\mathrm{k}_{\mu_Q(e_n)},\mathrm{k}^{\mathrm{n}}_{\mu_Q(e_n)}\right)\right\}$ is convergent to $\bar{a}$ that is $\lim_{n\to\infty}\tilde{\tilde{P}}\left(\mathrm{k^n}_{\mu_Q(e_\mathrm{n})},\mathrm{k^n}_{\mu Q(e_\mathrm{n})}\right)=\bar{a}$ since the $\mathcal{F}_{\mathrm{so}}$ -sequence $\left\{\tilde{\tilde{P}}\left(\mathrm{k^n}_{\mu_Q(e_\mathrm{n})},\mathrm{k^n}_{\mu Q(e_\mathrm{n})}\right)\right\}$ has boundaries in $\mathrm{R}(W)$.

Now the proof that $\left\{\tilde{\tilde{P}}\left(\mathrm{k^n}_{\mu_\ell(e_\mathrm{n})},\mathrm{k^n}_{\mathrm{QQ}(e_\mathrm{n})}\right)\right\}$ is a $\mathcal{F}_{\mathrm{so}}$ -Cauchy sequence in $\mathrm{R}(W)$ has to be done. As $\left\{k^n{}_{\mu_{Q(e_n)}}\right\}$ is a $\mathcal{F}_{\mathrm{so}}$ − Cauchy sequence in $\left(\mathcal{F}_{\mathrm{so}}\left(\mathcal{K}\right),\tilde{\mathrm{D}}\right)$, for every $\varepsilon > 0$, there is $k_\varepsilon \in \mathbb{N}$ such that for every $\mathrm{n},\mathrm{m} \geq k_\varepsilon$,$\tilde{\mathrm{D}}\left(\mathrm{k^m}_{\mu_\mathrm{Q}(e_\mathrm{m})},\mathrm{k^n}_{\mu_\mathrm{Q}(e_\mathrm{n})}\right) < \frac{s}{2}$.

Consequently, for every n, m $\geq k_\varepsilon$,

$$\left|\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k^n}_{\mu_Q(e_n)},\mathrm{k^n}_{\mu_Q(e_n)}\right)-\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k^m}_{\mu_Q(e_m)},\mathrm{k^m}_{\mu_Q(e_m)}\right)\right|=\qquad \tilde{\tilde{D}}\left(\mathrm{k^n}_{\mu_Q(e_n)},\mathrm{k^m}_{\mu_Q(e_m)}\right)\,|\leq \tilde{2}\tilde{D}\left(\mathrm{k^m}_{\mu_Q(e_m)},\mathrm{k^n}_{\mu_Q(e_n)}\right)<\varepsilon$$

because of

$$\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k^n}_{\mu_Q(e_\mathrm{n})},\mathrm{k^n}_{\mu_Q(e_\mathrm{n})}\right)=\tilde{\tilde{D}}\left(\mathrm{k^m}_{\mu_Q(e_\mathrm{m})},\mathrm{k^n}_{\mu_Q(e_\mathrm{n})}\right)+\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k^m}_{\mu_Q(e_\mathrm{m})},\mathrm{k^m}_{\mu_Q(e_\mathrm{m})}\right)-\tilde{\tilde{D}}\left(\mathrm{k^n}_{\mu_Q(e_\mathrm{n})},\mathrm{k^m}_{\mu_Q(e_\mathrm{m})}\right)$$

Therefore $\lim_{n\to\infty}\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k^n}_{\mu_Q(e_n)},\mathrm{k^n}_{\mu_Q(e_n)}\right)=\bar{a}$. On the other hand,

$$\begin{aligned}\left|\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k^n}_{\mu_Q(e_n)},\mathrm{m^m}_{\mu_Q(e_m)}\right)-a\right| &= \left|\tilde{\tilde{P}}\left(\mathrm{k^n}_{\mu_Q(e_n)},\mathrm{m^m}_{\mu_Q(e_\mathrm{m})}\right)-\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k^n}_{\mu_Q(e_n)},\mathrm{k^n}_{\mu_Q(e_n)}\right)+\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k^n}_{\mu_Q(e_n)},\mathrm{k^n}_{\mu_Q(e_n)}\right)-a\right|\\ &\leq \tilde{\tilde{D}}\left(\mathrm{k^n}_{\mu_Q(e_n)},\mathrm{k^m}_{\mu_Q(e_m)}\right)+\left|\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k^n}_{\mathrm{Q}(e_\mathrm{E})},\mathrm{k^n}_{\mu_{Q(e_n)}}\right)-\bar{a}\right|\end{aligned}$$

for every $n,m \geq k_\varepsilon$.

Thus, $\lim_{n,m\to\infty}\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k^n}_{\mu_Q(e_\mathrm{n})},\mathrm{k^m}_{\mu_{Q(e_\mathrm{m})}}\right)=\bar{a}$ and $\left\{k^n{}_{\mu_{Q(e_n)}}\right\}$ is a $\mathcal{F}_{\mathrm{so}}$ − Cauchy sequence in $\left(\mathcal{F}_{\mathrm{so}}\left(\mathcal{K}\right),\tilde{\tilde{\mathcal{P}}}\right)$.

(2) Let $\left(\mathcal{F}_{\mathrm{so}}\left(\mathcal{K}\right),\tilde{\tilde{\mathcal{P}}}\right)$ be complete a $\mathcal{F}_{\mathrm{so}}$ $-PMS$. Let $\left\{k^n{}_{\mu_{Q(e_n)}}\right\}$ be a $\mathcal{F}_{\mathrm{so}}$ − Cauchy sequence in $\left(\mathcal{F}_{\mathrm{so}}\left(\mathcal{K}\right),\tilde{\tilde{\mathcal{D}}}\right)$. Then from part (1) We get $\left\{k^n{}_{\mu_{Q(e_n)}}\right\}$ is a $\mathcal{F}_{\mathrm{so}}$ − Cauchy sequence in $\left(\mathcal{F}_{\mathrm{so}}\left(\mathcal{K}\right),\tilde{\tilde{\mathcal{P}}}\right)$. Thus, it converges to a $\mathcal{F}_{\mathrm{so}}$ − point at $\mathrm{k}^0{}_{\mu_{\mathrm{Q}(e_0)}\in\mathcal{F}_{\mathrm{so}}(\mathcal{K})}$ with

$$\lim_{n,m\to\infty}\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k^n}_{\mu_Q(e_\mathrm{n})},\mathrm{k^m}_{\mu_Q(e_\mathrm{m})}\right)=\lim_{n\to\infty}\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^0_{\mu_Q(e_0)},\mathrm{k}^\mathrm{n}_{\mu_Q(e_\mathrm{n})}\right)=\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^0_{\mu_Q(e_0)},\mathrm{k}^0_{\mu_Q(e_0)}\right).$$

Given $\varepsilon > 0$, there exists $k_{\varepsilon} \in \mathbb{N}$ such that $\tilde{\tilde{p}}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{Q}(\mathrm{e_n})}\right) - \tilde{\mathcal{P}}^{\tilde{(}}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}, \mathrm{k}^{0}{}_{\mu_{Q}(e_0)}\right) < \varepsilon$ and $\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}, \mathrm{k}^{0}{}_{\mu_{Q}(e_0)}\right)) - \tilde{\mathcal{P}}\left(\mathrm{k}^{\mathrm{n}}{}_{\mu_{\mathrm{Q}(e_\mathrm{n})}}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{\mathrm{Q}(e_\mathrm{n})}}\right) < \varepsilon$ when $n \geq k_{\varepsilon}$.

So we get

$$\tilde{\tilde{D}}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{Q}(e_\mathrm{n})}\right) = \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{\mathrm{Q}(e_\mathrm{e})}}\right) - \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{0}_{\mu_{Q}(e_0)}, \mathrm{k}^{0}{}_{\mu_{Q}(e_0)}\right) < \varepsilon,$$

and

$$\begin{aligned}\tilde{\tilde{D}}\left(\mathrm{k}^{\mathrm{n}}{}_{\mu_{Q}(e_n)}, \mathrm{k}^{0}{}_{\mu_{Q}(e_0)}\right) =& \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{Q}(e_n)}\right) - \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{\mathrm{n}}{}_{\mu_{Q}(e_\mathrm{n})}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{Q}(e_\mathrm{n})}\right) \leq \left|\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{Q}(e_\mathrm{E})}\right) - \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}_{\mu_{Q}(e_0)}, \mathrm{k}_{\mu_{Q}(e_0)}\right)\right| \\ &+ \left|\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}, \mathrm{k}^{0}{}_{\mu_{\ell(e_0)}}\right) - \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{\mathrm{n}}{}_{\mu_{\mathrm{Q}(e_\mathrm{n})}}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{Q}(e_\mathrm{n})}\right)\right| \leq 2\varepsilon \quad \text{when} \quad n \geq k_{\varepsilon}.\end{aligned}$$

So $\left(\mathcal{F}_{\mathrm{so}}(\mathcal{K}), \tilde{\tilde{D}}\right)$ is complete. Finally, verifying that

$$\lim_{n\to\infty} \tilde{\tilde{D}}\left(\bar{a}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{\mathrm{Q(en_n)}}}\right) = \bar{0} \quad \text{if and only if} \quad \tilde{\tilde{\mathcal{P}}}(\bar{a}, \bar{a}) = \lim_{n\to\infty} \tilde{\tilde{\mathcal{P}}}\left(\bar{a}, \mathrm{k}^{\mathrm{n}}{}_{\mu\mathrm{Q}(e_\mathrm{n})}\right) = \lim_{n,m\to\infty} \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{\mathrm{n}}{}_{\mu\mathrm{Q}(e_\mathrm{n})}, \mathrm{k}^{\mathrm{m}}{}_{\mu\mathrm{Q}(e_\mathrm{m})}\right)$$

Let $\lim_{n\to\infty} \tilde{\tilde{D}}\left(\bar{a}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{Q}(\mathrm{e}_n)}\right) = \bar{0}$. To prove $\tilde{\tilde{\mathcal{P}}}(\bar{a}, \bar{a}) = \lim_{n\to\infty} \tilde{\mathcal{P}}\left(\bar{a}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{Q}(\mathrm{e_n})}\right) = \lim_{n,m\to\infty} \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{\mathrm{n}}{}_{\mu_{Q}(e_\mathrm{n})}, \mathrm{k}^{\mathrm{m}}{}_{\mu_{Q}(e_\mathrm{m})}\right)$. By use (1,1) we get $\lim_{n\to\infty} \tilde{\tilde{\mathrm{D}}}\left(\bar{a}, \mathrm{k}^{\mathrm{n}}{}_{\mu\mathrm{Q}(e_\mathrm{n})}\right) = \lim_{n\to\infty}\left[\tilde{\tilde{\mathcal{P}}}\left(\bar{a}, \mathrm{k}^{\mathrm{n}}{}_{\mu\mathrm{Q}(e_n)}\right) - \tilde{\tilde{\mathcal{P}}}(\bar{a}, \bar{a})\right] = \bar{0}$ from above we get $\lim_{n\to\infty} \tilde{\tilde{\mathcal{P}}}\left(\bar{a}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{\mathrm{Q(e_n)}}})\right) = \tilde{\tilde{\mathcal{P}}}(\bar{a}, \bar{a})$ since $\tilde{\tilde{\mathcal{P}}}(\bar{a}, \bar{a}) = \lim_{n,m\to\infty} \tilde{\tilde{P}}\left(\mathrm{k}^{\mathrm{n}}{}_{\mu\mathrm{Q}(e_\mathrm{n})}, \mathrm{k}^{\mathrm{m}}{}_{\mu\mathrm{Q}(e_\mathrm{m})}\right)$ then we have $\tilde{\tilde{\mathcal{P}}}(\bar{a}, \bar{a}) = \lim_{n\to\infty} \tilde{\tilde{\mathcal{P}}}\left(\bar{a}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{\mathrm{Q}}(e_\mathrm{n})}\right) = \lim_{n,m\to\infty} \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{\mathrm{n}}{}_{\mu_{\mathrm{Q}}(e_\mathrm{n})}, \mathrm{k}^{\mathrm{m}}{}_{\mu_{\mathrm{Q}}(e_\mathrm{m})}\right)$.

Now let $\tilde{\tilde{\mathcal{P}}}(\bar{a}, \bar{a}) = \lim_{n\to\infty} \tilde{\tilde{\mathcal{P}}}\left(\bar{a}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{\mathrm{Q}}(e_n)}\right) = \lim_{n,m\to\infty} \tilde{\tilde{\mathcal{P}}}^{\tilde{n}}\left(\mathrm{k}^{\mathrm{n}}{}_{\mu_{\mathrm{Q}}(e_n)}, \mathrm{k}^{\mathrm{m}}{}_{\mu_{\mathrm{Q}}(e_\mathrm{m})}\right)$ to prove $\lim_{n\to\infty} \tilde{\tilde{D}}\left(\bar{a}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{\mathrm{Q(e_n)}}}\right) = \bar{0}$. Since $\tilde{\mathcal{P}}(\bar{a}, \bar{a}) = \lim_{n\to\infty} \tilde{\mathcal{P}}\left(\bar{a}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{\mathrm{Q(e_n)}}})\right)$ then we get $\lim_{n\to\infty}\left[\tilde{\mathcal{P}}\left(\bar{a}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{\mathrm{Q(e_n)}}})\right) - \tilde{\mathcal{P}}(\bar{a}, \bar{a})\right] = \bar{0}$ By use (1,1) we get $\lim_{n\to\infty} \tilde{\tilde{D}}\left(\bar{a}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{\mathrm{Q(e_n)}}}\right) = \bar{0}$.

Theorem 3.7 Let $\left(\mathcal{F}_{\mathrm{so}}\left(\mathcal{K}\right), \tilde{\tilde{\mathcal{P}}}\right)$ be complete a $\mathcal{F}_{\mathrm{so}} - PMS$, and $\mathrm{T} : \left(\mathcal{F}_{\mathrm{so}}\left(\mathcal{K}\right), \tilde{\tilde{\mathcal{P}}}\right) \rightarrow \left(\mathcal{F}_{\mathrm{so}}\left(\mathcal{K}\right), \tilde{\mathcal{P}}\right)$ $\mathcal{F}_{\mathrm{so}}$ -mapping such that there exist $r \in \mathrm{R}^{+}(W)$ with $\bar{0} < \bar{r} < \bar{1}$, satisfying

$$\left|\tilde{\tilde{\mathcal{P}}}\left(\mathrm{T}\psi\left(\mathrm{k}^{1}{}_{\mu_{\mathrm{Q}(\mathrm{e}_1)}}\right), \mathrm{T}\psi\left(\mathrm{k}^{2}{}_{\mu_{\mathrm{Q}(\mathrm{e}_2)}}\right)\right)\right| \leq \bar{r}\left|\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{1}{}_{\mu_{\mathrm{Q}(\mathrm{e}_1)}}, \mathrm{k}^{2}{}_{\mu_{\mathrm{Q}(\mathrm{e}_2)}}\right)\right|, \tag{1.3}$$

For every $\mathrm{k}^{1}{}_{\mu_{\mathrm{Q}(e_1)}}, \mathrm{k}^{2}{}_{\mu_{\mathrm{Q}(\mathrm{e}_2)}} \tilde{\mathrm{E}}\mathcal{F}_{\mathrm{so}}(\mathcal{K})$. Then T has a unique $\mathcal{F}_{\mathrm{so}}$ - fixed point. Proof. Put $\mathrm{k}^{0}{}_{\mu_{\mathrm{Q}(\mathrm{e}_0)}} \tilde{\in} \mathcal{F}_{\mathrm{so}}(\mathcal{K})$.

Then it's clear that for any $n \in \mathbb{N}$

$$\left|\tilde{\tilde{\mathcal{P}}}\left(T^{n}\left(k^{0}_{\mu_{Q}(e_0)}\right), T^{n}\left(k^{0}_{\mu_{Q}(e_0)}\right)\right)\right| \leq r^{n}\left|\tilde{\tilde{\mathcal{P}}}\left(k^{0}_{\mu_{Q}(e_0)}, k^{0}_{\mu_{Q}(e_0)}\right)\right|$$

And

$$\left|\tilde{\tilde{\mathcal{P}}}\left(T^{n}\left(k^{0}_{\mu_{Q}(e_0)}\right), T^{n+1}\left(k^{0}_{\mu_{Q}(e_0)}\right)\right)\right| \leq r^{n}\left|\tilde{\tilde{\mathcal{P}}}\left(k^{0}_{\mu_{Q}(e_0)}, T(k^{0}_{\mu_{Q}(e_0)})\right)\right|$$

By (1,1)

$$\tilde{\tilde{D}}\left(T^{n}\left(k^{0}_{\mu_{Q}(e_0)}\right), T^{n+1}\left(k^{0}_{\mu_{Q}(e_0)}\right)\right) + \tilde{\tilde{P}}\left(T^{n}\left(k^{0}_{\mu_{Q}(e_0)}\right), T^{n}\left(k^{0}_{\mu_{Q}(e_0)}\right)\right) = \tilde{\tilde{P}}\left(T^{n}\left(k^{0}_{\mu_{Q}(e_0)}\right), T^{n+1}\left(k^{0}_{\mu_{Q}(e_0)}\right)\right)$$

We conclude that

$$\tilde{\tilde{D}}\left(T^{n}\left(k^{0}_{\mu_{Q}(e_0)}\right), T^{n+1}\left(k^{0}_{\mu_{Q}(e_0)}\right)\right) + \tilde{\tilde{P}}\left(T^{n}\left(k^{0}_{\mu_{Q}(e_0)}\right), T^{n}\left(k^{0}_{\mu_{Q}(e_0)}\right)\right) \leq r^{n}\left|\tilde{\tilde{P}}\left(k^{0}_{\mu_{Q}(e_0)}, T\left(k^{0}_{\mu_{Q}(e_0)}\right)\right)\right|$$

$$\begin{aligned}\text{Hence} \quad \tilde{\tilde{D}}\left(\mathrm{T}^{n}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}\right), \mathrm{T}^{n+1}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}\right)\right) &\leq r^{n}\left|\tilde{\mathcal{P}}^{\tilde{(}}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}, T\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}\right)\right)\right| \tilde{\tilde{P}}\left(T^{n}\left(k^{0}_{H_{Q}(a_0)}\right), T^{n}\left(k^{0}_{H_{Q}(a_0)}\right)\right) \\ &\leq r^{n}\left|\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}, \mathrm{T}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}\right)\right)\right| + \left|\tilde{\tilde{\mathcal{P}}}\left(\mathrm{T}^{n}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}\right), \mathrm{T}^{n}\left(\mathrm{k}^{0}{}_{\mu_{Q(e_0)}}\right)\right)\right| \\ &\leq r^{n}\left[\left|\tilde{\tilde{P}}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}, \mathrm{T}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}\right)\right)\right| + \mid \tilde{\tilde{P}}\left(\mathrm{k}^{0}{}_{\mu_{Q}(e_0)}, \mathrm{k}^{0}{}_{Q(e_0)}\right) \|\right].\end{aligned}$$

Now let $n, k \in \mathbb{N}$

$$\begin{aligned}\text{Then}\quad & \tilde{D}\left(T^n\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right), \mathrm{T}^{n+k}\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right)\right)\\ & \leq \tilde{\tilde{D}}\left(T^n\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right), \mathrm{T}^{n+1}\left(\mathrm{k}^0_{\mu}(e_0)\right)\right) + \cdots + \tilde{D}\left(T^{n+k-1}\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right), \mathrm{T}^{n+k}\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right)\right)\\ & \leq \left(r^n + \cdots + r^{n+k-1}\right)\left[\left|\tilde{\tilde{P}}\left(\mathrm{k}^0{}_{\mu_Q(e_0)}, \mathrm{T}\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right)\right)\right| + \left|\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^0{}_{\mu_Q(e_0)}, \mathrm{k}^0{}_{\mu_{Q(e_0)}}\right)\right|\right]\\ & \leq \frac{r^n}{\bar{1} - r}\left[\left|\tilde{\tilde{P}}\left(\mathrm{k}^0{}_{\mu_Q(e_0)}, T\left(\mathrm{k}^0{}_{\mathrm{H}_Q(e_0)}\right)\right)\right| + \left|\tilde{\mathcal{P}}\left(\mathrm{k}^0{}_{\mathrm{H}_Q(e_0)}, \mathrm{k}^0{}_{\mu_Q(e_0)}\right)\right|\right].\end{aligned}$$

Similarly, we get that.

$$\tilde{\tilde{D}}\left(\mathrm{T}^{n+k}\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right), T^n\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right)\right) \leq \frac{r^n}{\hat{1} - r}\left\|\tilde{\mathcal{P}}^2\left(\mathrm{k}^0{}_{\mu_Q(e_0)}, T\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right)\right)\right| + |\tilde{\mathcal{P}}^(\left(\mathrm{k}^0{}_{\mu_Q(e_0)}, \mathrm{k}^0{}_{\mu_Q(e_0)}\right)\right\|.$$

As a result, $\left\{\mathrm{T}^n\left(\mathrm{k}^0{}_{\mu_{Q(e_0)}}\right)\right\}$ is a complete Cauchy sequence in the $\mathcal{F}_{\text{so}}$ -metric space $(\mathcal{F}_{\text{so}}(\mathcal{K}), \tilde{\mathrm{D}})$ as shown in Lemma (3.6). There is, therefore, $\mathrm{k}^{\mathrm{x}}{}_{\mathrm{H}_{Q(\mathrm{ex_x}}} \tilde{\mathrm{E}}\mathcal{F}_{\text{so}}(\mathcal{K})$ such that $\lim_{n\to\infty} \tilde{\tilde{D}}\left(\mathrm{k}^{\mathrm{x}}{}_{\mu_Q(e_x)}, \mathrm{k}^{\mathrm{n}}{}_{\mu_{Q(e_n)})}\right) = \bar{0}$. Our goal is to demonstrate that $\mathrm{k}^{\mathrm{x}}{}_{\mu_Q(e_x)}$ is unique fixed point of T .

First, notice that, by(2) in Lemma (3.6), we get

$$\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{\mathrm{x}}{}_{\mu_Q(e_z)}, \mathrm{k}^{\mathrm{x}}{}_{\mu_Q(e_z)}\right) = \lim_{\mathrm{n}\to\infty} \tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{\mathrm{x}}{}_{\mu_Q(e_z)}, \mathrm{T}^{\mathrm{n}}\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right)\right) = \lim_{n,m\to\infty} \tilde{\tilde{\mathcal{P}}}\left(\mathrm{T}^{\mathrm{n}}\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right), \mathrm{T}^m\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right)\right)$$

Also, since

$$\lim_{n,m\to\infty} \tilde{\tilde{D}}\left(\mathrm{T}^n\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right), \mathrm{T}^m\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right)\right) = \lim_{n\to\infty} \tilde{\tilde{\mathcal{P}}}\left(\mathrm{T}^n\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right), \mathrm{T}^n\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right)\right) = \bar{0},$$

we conclude from (1.1) that

$$\lim_{n,m\to\infty} \tilde{\tilde{\mathcal{P}}}\left(\mathrm{T}^n\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right), \mathrm{T}^m\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right)\right) = \bar{0}.$$

As a result

$$\tilde{\mathcal{P}}\left(\mathrm{k}^{\mathrm{x}}{}_{\mathrm{H}_Q(e_{\mathrm{x}})}, \mathrm{k}^{\mathrm{x}}{}_{\mu_{Q(e_x)}}\right) = \lim_{n\to\infty} \tilde{\mathcal{P}}\left(\mathrm{k}^{\mathrm{x}}{}_{\mu_{Q(e_e)}}, \mathrm{T}^n\left(\mathrm{k}^0{}_{\mu_{Q(e_0)}}\right)\right) = \bar{0}.$$

Now, since

$$\left|\tilde{\mathcal{P}}\left(\mathrm{T}\left(\mathrm{k}^{\mathrm{x}}{}_{\mu_Q(e_x)}\right), \mathrm{T}\left(\mathrm{k}^{\mathrm{x}}{}_{\mu_Q(e_x)}\right)\right)\right| \leq r\left|\tilde{\mathcal{P}}\left(\mathrm{k}^{\mathrm{x}}{}_{\mu_Q(e_x)}, \mathrm{k}^{\mathrm{x}}{}_{\mu_{Q(e_x)}}\right)\right| = \bar{0},$$

It follows that $\tilde{\tilde{\mathcal{P}}}\left(\mathrm{T}\left(\mathrm{k}^{\mathrm{x}}{}_{\mu_Q(\mathrm{e_X})}\right), \mathrm{T}\left(\mathrm{k}^{\mathrm{x}}{}_{\mu_Q(\mathrm{e_X})}\right)\right) = \bar{0}$.

On the other hand, since

$$\left|\tilde{\mathcal{P}}\left(\mathrm{T}\left(\mathrm{k}^{\mathrm{x}}{}_{\text{-}_{\mathrm{Q}(\mathrm{e_x}))}}\right), \mathrm{T}^{n+1}\left(\mathrm{k}^{\mathrm{x}}{}_{\text{-}_{\mathrm{Q}(\mathrm{e_x}))}}\right)\right)\right| \leq r\left|\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{\mathrm{x}}{}_{\text{-}_{Q(e_x)}}, \mathrm{T}^n\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right)\right)\right|,$$

thus, it follows that

$$\lim_{n\to\infty} \tilde{\tilde{\mathcal{P}}}\left(\mathrm{T}\left(\mathrm{k}^{x}{}_{\mu_Q(e_x)}\right), \mathrm{T}^n\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right)\right) = \bar{0}.$$

Then Lemma (3.6) demonstrates that $\mathrm{T}\left(\mathrm{k}^{\mathrm{x}}{}_{\text{-}_{\mathrm{Q}(\mathrm{e_x})}}\right)$ is the limit point of $\left\{\mathrm{T}^n\left(\mathrm{k}^0{}_{\mu_Q(e_0)}\right)\right\}$ in $(\mathcal{F}_{\text{so}}(\mathcal{K}), \tilde{\mathrm{D}})$. Therefore, $\mathrm{T}\left(\mathrm{k}^{\mathrm{x}}{}_{\mu_{\mathrm{Q}}(\mathrm{e_x})}\right) = \mathrm{k}^{\mathrm{x}}{}_{\mu_{Q(e_x)}}$. Finally, let $\mathrm{k}^{\mathrm{y}}{}_{\mu_{Q(e_y)}} \tilde{\in} \mathcal{F}_{\text{so}}(\mathcal{K})$ such that $T\left(\mathrm{k}^{y}{}_{\mathrm{Q}_Q(e_{\mathrm{y}})}\right) = \mathrm{k}^{\mathrm{y}}{}_{\mathrm{Q}_Q(\mathrm{e_y})}$.

Then

$$\left|\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{\mathrm{x}}{}_{\mu_{\mathrm{Q}}(\mathrm{e_x})}, \mathrm{k}^{\mathrm{y}}{}_{\mu_{\mathrm{Q}}(\mathrm{e_y})}\right)\right| = \left|\tilde{\tilde{\mathcal{P}}}\left(\mathrm{T}\left(\mathrm{k}^{\mathrm{x}}{}_{\mu_{\mathrm{Q}}(\mathrm{e_x})}\right), \mathrm{T}\left(\mathrm{k}^{\mathrm{y}}{}_{\mu_{\mathrm{Q}}(\mathrm{e_y})}\right)\right)\right| \leq r\left|\tilde{\tilde{\mathcal{P}}}\left(\mathrm{k}^{\mathrm{x}}{}_{\mu_{\mathrm{Q}}(\mathrm{e_x})}, \mathrm{k}^{\mathrm{y}}{}_{\mu_{\mathrm{Q}}(\mathrm{e_y})}\right)\right|.$$

which implies that $\mathrm{k}^{\mathrm{y}}{}_{\mathrm{H}_{Q(e_y)}} = \mathrm{k}^{\mathrm{x}}{}_{Q(e_x)}$.

4. Conclusions

We defined a partial fuzzy soft metric space and proved that the fuzzy soft fixed point is unique in this space. Researchers can also study fuzzy soft strong partial metric space.

Acknowledgements

We thank Nineveh Education Directorate, Ministry of Education.

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