New Class Of Generalized Convex Functions And Its Applications To Optimization Problems

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In this paper, a new class of generalized convex function called the class of exponentially E-preinvex functions is introduced by combining the classes of exponentially E-convex functions and exponentially preinvex functions. Some important properties and relations are proved, especially those which relate exponentially E-preinvex functions with different γ-level sets and different epigraphs associated with these functions. Also, some optimality properties for nonlinear optimization problems involving exponentially E-preinvex functions are established.

Keywords: E-invex set, preinvex function, exponentially preinvex function, exponentially E-preinvex function.

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1. Introduction and preliminaries

The classical theory of convexity plays an essential role in various fields of science such as mathematics, engineering, economics and physics. This is due to its theoretical [1–4] and practical importance as a powerful tool for the development of numerical methods to solve practical problems, especially in mathematical optimization problems [5, 6]. However, imposing convexity conditions on sets and functions in mathematical optimization problems entails some limitations in the applications [7]. For this reason, and for theoretical interest, various extensions and generalizations close to convexity in some sense have been developed and investigated by relaxing the convexity assumptions using a variety of ideas and techniques. For example, in [8], a class of functions called convexlike and a saddle point optimality condition for convexlike mathematical programs was established. Hanson introduced a new class of differentiable functions in 1981 to weaken the convexity condition, so that the Kuhn-Tucker conditions of optimality for minima became sufficient [7]. The new class introduced in [7] is named as invex functions by Craven [9, 10]. The more general class of functions (not necessarily differentiable), called preinvex functions, was developed by Ben Israel and Mond [11] and Hanson and Mond [12]. These functions are used to develop some constraint qualifications and duality results for scalar-valued optimization problem [13]. As a generalization of convex sets, the concept of invex sets was first formally defined on subsets of real numbers [13] and then extended to subsets of n-dimensional space [14]. Preinvex functions are usually defined on invex sets. For more recent work on invex sets, preinvex functions and generalized preinvex functions and their applications in applied mathematics and optimization theory (see e.g. [15–20]). In recent years, new important generalizations of convex functions were developed [21–23].

Another important generalization of convex sets and convex functions, named as E-convex sets and E-convex functions, was introduced by Youness [24], where a mapping $E : \mathbb{R}^n \to \mathbb{R}^n$ has the main effect in the study of this kind of generalization. Youness investigated many important properties of E-convexity and was the first to define an E-convex optimization problem and develop its stability and optimality properties [25, 26]. Youness and his collaborators also developed some duality properties in E-convex optimization problem and studied optimality conditions for E-convex problem that have E-differentiable objective functions [27, 28]. This work motivated many researchers to further study E-convexity and develop many generalizations and extensions of E-convex functions and their applications in single objective and multiobjective optimization problems (see, e.g., [29–38] and the references therein).

Fulga and Perda [39] studied E-preinvex functions defined on E-invex sets as a new generalization of convexity by combining invex sets (resp. preinvex functions) with E-convex sets (resp. E-convex functions). Inspired by [39], many researchers have introduced and studied new generalized convex functions by combining E-convex functions with some generalizations of preinvex functions (see, e.g., [40–44]).

The class of exponentially convex functions is regarded as a further extension of convex functions. The importance of this class arises from its applications in information theory, data analysis, machine learning and statistics [45, 46]. Recent research on the properties of exponential convex functions and their applications in mathematics and optimization problems can be found in the papers [47–52]. In 2019, Noor and Noor introduced the class of exponentially preinvex functions and derived some interesting general and optimality properties that this class possesses [53]. More recently, Abdulaleem [54] introduced the class of exponentially E-convex functions as a generalization of the class of exponentially convex functions. Abdulaleem used this class in the development of some optimality conditions for multiobjective programming problems.

Following the ongoing research on generalized convexity, our aim in this paper is to introduce the class of exponentially E-preinvex functions and to study some of their general and optimality properties. For rest of this section, some preliminary concepts related to our work are recalled and new definitions needed in this work (see Definition 1.9 and Definition 1.10(2-3)). In section two the class of exponentially E-preinvex functions is defined, by combining exponentially E-convex functions with exponentially preinvex functions, and two examples of exponentially and non exponentially E-preinvex functions are shown. Then, some general properties are discussed and different necessary and sufficient conditions for a function f to be exponentially E-preinvex using γ-level sets and the epigraph sets of f are proved. In section three, a nonlinear optimization problem is constructed by employing exponentially E-preinvex functions and new results and optimality properties are deduced.

Throughout this paper, let $A \subseteq \mathbb{R}^n$ be a non-empty set and $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$.

Assume also that $E : \mathbb{R}^n \to \mathbb{R}^n$ where $E(x)$ is written as Ex and $\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are two given mappings.

The following needed preliminary concepts are recalled.

Definition 1.1. [14] The set A is called invex if for each $a_1, a_2 \in A$ and for each $\lambda \in [0,1]$, we have $a_2 + \lambda\omega(a_1, a_2) \in A$.

Definition 1.2. [39] Let A_1 and A_2 be two subsets of $\mathbb{R}^n$. Then, A_1 is named as be slack invex (for short, s. invex) with respect to A_2 if, for each $a_1, a_2 \in A_1 \cap A_2$ and every $\lambda \in [0,1]$ such that $a_2 + \lambda\omega(a_1, a_2) \in A_2$ we get $a_2 + \lambda\omega(a_1, a_2) \in A_1$.

Definition 1.3. [24] The set A is said to be an E-convex set if $\lambda Ea_1 + (1-\lambda)Ea_2 \in A$, for each $a_1, a_2 \in A$ and $\lambda \in [0,1]$.

Definition 1.4. [39] The set A is said to be an E-invex set with respect to ω if $Ea_2 + \lambda\omega(Ea_1, Ea_2) \in A$, for each $a_1, a_2 \in A$ and $\lambda \in [0,1]$.

Proposition 1.5. [39] If A is E-invex. Then, $E(A) \subseteq A$.

Definition 1.6. [35] The function f is named exponentially preinvex on the invex set A, if for every $a_1, a_2 \in A$ and $\lambda \in [0,1]$

$$e^{f(a_2+\lambda\omega(a_1,a_2))} \leq \lambda e^{f(a_1)} + (1-\lambda)e^{f(a_2)}$$

Definition 1.7. [54] The function f is named exponentially E-convex on the E-convex set A, if for every $a_1, a_2 \in A$ and $\lambda \in [0,1]$

$$e^{f(\lambda Ea_1+(1-\lambda)Ea_2)} \leq \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)}$$

Definition 1.8. [53] The epigraph set of exponentially preinvex function f is given as

$$\text{epif} = \left\{(a,\gamma) \in A \times \mathbb{R} : e^{f(a)} \leq \gamma\right\}.$$

In a similar manner, we define the epigraphs associated with the mapping E as follows.

Definition 1.9.

$$E - epi\ f = \left\{(a,\gamma) \in A \times \mathbb{R} : e^{f(Ea)} \leq \gamma\right\}.$$

$$epi_E\ f = \left\{(Ea,\gamma) \in E(A) \times \mathbb{R} : e^{f(Ea)} \leq \gamma\right\}.$$

With each epigraph defined above, γ-level sets are associated, respectively, as follows.

Definition 1.10. Let $\gamma \in \mathbb{R}$. Then,

1. $H_\gamma = \left\{a \in A : e^{f(a)} \leq \gamma\right\}$.[53]
2. $E - H_\gamma = \left\{a \in A : e^{f(Ea)} \leq \gamma\right\}$.
3. $H_{\gamma,E} = \left\{Ea \in E(A) : e^{f(Ea)} \leq \gamma\right\}$.

2. Exponentially E-preinvex functions

In this section we define exponentially E-preinvex function and provide two different examples of exponentially and non exponentially E-preinvex functions. Then, we provide different properties of the new defined function which include general properties as well as providing different necessary and sufficient conditions to obtain exponentially E-preinvex function f in terms of γ-level sets and the epigraph sets of f.

Definition 2.1. The function f is named as exponentially E-preinvex on the E-invex set A, if for every $a_1, a_2 \in A$ and $\lambda \in [0,1]$

$$e^{f(Ea_2+\lambda\omega(Ea_1,Ea_2))} \leq \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)}.$$

The next example shows an exponentially E-preinvex function that is not exponentially preinvex.

Example 2.2. If $f, E : \mathbb{R} \to \mathbb{R}$ and $\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are shown below.

$$f(a) = \begin{cases} 4a & a > 0 \\ a^2+1 & a \leq 0 \end{cases}, \quad Ea = \begin{cases} \frac{(1+a)^2}{4} & a > 0 \\ \frac{(1-a)^2}{4} & a \leq 0 \end{cases}$$

and

$$\omega(a_1, a_2) = \begin{cases} a_1 - a_2 & a_1, a_2 > 0 \text{ or } a_1, a_2 \leq 0 \\ -a_1 & \text{otherwise} \end{cases}$$

for all $a_1, a_2 \in \mathbb{R}$. Note that f is exponentially E-preinvex on $\mathbb{R}$ but not exponentially preinvex function. Take, take $a_1 = 1, a_2 = -1$ and $\lambda = 1$. Then $\omega(a_1, a_2) = -1$. Hence, $e^{f(a_2+\lambda\omega(a_1,a_2))} = e^{f(-2)} = e^5$ and $\lambda e^{f(a_1)} + (1-\lambda)e^{f(a_2)} = e^4$.

This shows,

$$e^{f(a_2+\lambda\omega(a_1,a_2))} > \lambda e^{f(a_1)} + (1-\lambda)e^{f(a_2)}.$$

Hence, f is not exponentially preinvex on $\mathbb{R}$. To show that f is exponentially E-preinvex, we have three possibilities:

Case (1): If $a_1, a_2 > 0$ then

$$\begin{aligned} & e^{f(Ea_2+\lambda\omega(Ea_1,Ea_2))} \\ &= e^{\frac{(1+a_2)^2}{4}+\lambda\left(\frac{(1+a_1)^2}{4}-\frac{(1+a_2)^2}{4}\right)} \\ &= e^{\lambda\frac{(1+a_1)^2}{4}+(1-\lambda)\frac{(1+a_2)^2}{4}} \leq \lambda e^{\frac{(1+a_1)^2}{4}} + (1-\lambda)e^{\frac{(1+a_2)^2}{4}} \\ &= \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)} \end{aligned}$$

The above inequality holds for any $p, q > 0$ and $\lambda \in [0,1]$.
i.e., $p^\lambda q^{1-\lambda} \leq \lambda p + (1-\lambda)q$.

Case (2): If $a_1, a_2 \leq 0$ then

$$\begin{aligned} & e^{f(Ea_2+\lambda\omega(Ea_1,Ea_2))} \\ &= e^{\lambda(1-a_1)^2+(1-\lambda)(1-a_2)^2} \leq \lambda e^{(1-a_1)^2} + (1-\lambda)e^{(1-a_2)^2} \\ &= \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)} \end{aligned}$$

Case (3): If $a_1 > 0, a_2 \leq 0$ then

$$\begin{aligned} & e^{f(Ea_2+\lambda\omega(Ea_1,Ea_2))} \\ &= e^{\lambda(1+a_1)^2+(1-\lambda)(1-a_2)^2} \leq \lambda e^{(1+a_1)^2} + (1-\lambda)e^{(1-a_2)^2} \\ &= \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)} \end{aligned}$$

From the three cases, we have

$$e^{f(Ea_2+\lambda\omega(Ea_1,Ea_2))} \leq \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)}$$

as required.

Another example of a function, defined on a proper set A, which is neither exponentially E-preinvex nor exponentially preinvex.

Example 2.3. Let $A = [-2,-1] \cup [1,2]$, $f : A \to \mathbb{R}$, $E : \mathbb{R} \to \mathbb{R}$, and $\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are defined as.

$$f(a) = \begin{cases} 1 & a \in [-2,0] \\ \frac{1}{2} & \text{otherwise} \end{cases}, \quad Ea = \begin{cases} a^2 & a \in [-1.4, 1.4] \\ -1 & \text{otherwise} \end{cases}$$

and

$$\omega(a_1, a_2) = \begin{cases} a_1 - a_2 & a_1, a_2 \geq 0 \text{ or } a_1, a_2 \leq 0 \\ -2 - a_2 & a_1 > 0, a_2 \leq 0 \text{ or } a_1 \geq 0, a_2 < 0 \\ 1 - a_2 & a_1 < 0, a_2 \geq 0 \text{ or } a_1 \leq 0, a_2 > 0 \end{cases}$$

From [Example 2.1,39], the set A is an invex and E-invex, respectively. However, the function f is neither exponentially E-preinvex nor exponentially preinvex on A with respect to ω. To show f is not exponentially E-preinvex, let $a_1 \in [-1.4, -1]$, $a_2 \in [-2, -1.4]$ then

$$\begin{aligned} & e^{f(Ea_2+\lambda\omega(Ea_1,Ea_2))} = e^{f(-1-\lambda)} \\ &= e^1 \geq \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)} = -1.1\lambda + e^1 \end{aligned}$$

Then, f is not exponentially preinvex.

Some properties of exponentially E-preinvex functions on $\mathbb{R}^n$ are given next.

Proposition 2.4. Assume that f is exponentially E-preinvex function on $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$ then $f + \alpha$ is exponentially E-preinvex function on $\mathbb{R}^n$.

Proof. Since f is exponentially E-preinvex function on $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$ then $e^\alpha > 0$ and for any $a_1, a_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$

$$e^{f(Ea_2+\lambda\omega(Ea_1,Ea_2))}e^\alpha \leq \lambda e^{f(Ea_1)}e^\alpha + (1-\lambda)e^{f(Ea_2)}e^\alpha.$$

i.e.,

$$e^{f(Ea_2+\lambda\omega(Ea_1,Ea_2))+\alpha} \leq \lambda e^{f(Ea_1)+\alpha} + (1-\lambda)e^{f(Ea_2)+\alpha}.$$

Thus, $f + \alpha$ is exponentially E-preinvex function on $\mathbb{R}^n$.

Proposition 2.5. Let f and g are two exponentially E-preinvex functions on $\mathbb{R}^n$ and $\alpha_1, \alpha_2 \geq 0$ then $e^h = \alpha_1 e^f + \alpha_2 e^g$ is exponentially E-preinvex function on $\mathbb{R}^n$.

Proof. From the assumptions on f and g then for any $a_1, a_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$

$$e^{f(Ea_2+\lambda\omega(Ea_1,Ea_2))} \leq \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)} \quad (1)$$

$$e^{g(Ea_2+\lambda\omega(Ea_1,Ea_2))} \leq \lambda e^{g(Ea_1)} + (1-\lambda)e^{g(Ea_2)} \quad (2)$$

Now, from the definition of e^h with Eqs. (1) and (2) we get

$$\begin{aligned}
&e^{h(Ea_2+\lambda\omega(Ea_1,Ea_2))} \\
&= \alpha_1 e^{f(Ea_2+\lambda\omega(Ea_1,Ea_2))} + \alpha_2 e^{g(Ea_2+\lambda\omega(Ea_1,Ea_2))} \leq \\
&\alpha_1\left(\lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)}\right) + \alpha_2\left(\lambda e^{g(Ea_1)} + (1-\lambda)e^{g(Ea_2)}\right) \\
&= \lambda\left(\alpha_1 e^{f(Ea_1)} + \alpha_2 e^{g(Ea_1)}\right) + (1-\lambda)\left(\alpha_1 e^{f(Ea_2)} + \alpha_2 e^{g(Ea_2)}\right) \\
&= \lambda e^{h(Ea_1)} + (1-\lambda)e^{h(Ea_2)}
\end{aligned}$$

Thus, e^h is exponentially E-preinvex function on $\mathbb{R}^n$.

Proposition 2.6. Let f and h are two real functions on $\mathbb{R}^n$ and $h = e^f$. Then f is exponentially E-preinvex function on $\mathbb{R}^n$ if and only if h is exponentially E-preinvex function on $\mathbb{R}^n$.

Proof. Suppose that f is exponentially E-preinvex function then for any $a_1, a_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$

$$e^{f(Ea_2+\lambda\omega(Ea_1,Ea_2))} \leq \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)}.$$

From the definition of h, we obtain the exponentially E-preinvexity of h.

i.e., $h(Ea_2 + \lambda\omega(Ea_1, Ea_2)) \leq \lambda h(Ea_1) + (1-\lambda)h(Ea_2)$. The reverse direction, of this property, can be obtained similarly.

Remark 2.7. For the rest of this section, the set A is an E-invex set and $E(A)$ is an invex set.

The next three propositions provide different conditions for f to be exponentially E-preinvex using the s. invexity, E-invexity, and invexity for the γ-level sets introduced in Definitions 1.10.

Proposition 2.8. Let f is exponentially E-preinvex on A. Then H_γ is s. invex set with respect to $E(A)$, for any $\gamma \in \mathbb{R}$.

Proof. Take $a_1, a_2 \in H_\gamma \cap E(A)$ and $\lambda \in [0,1]$ such that $a_1, a_2 \in E(A), e^{f(a_1)} \leq \gamma$,

$$e^{f(a_2)} \leq \gamma, \text{ and } \quad a_2 + \lambda\omega(a_1, a_2) \in E(A) \subseteq A \quad (3)$$

Since f is exponentially E-preinvex on A, then

$$\begin{aligned}
e^{f(a_2+\lambda\omega(a_1,a_2))} &\leq \lambda e^{f(a_1)} + (1-\lambda)e^{f(a_2)} \\
&\leq \lambda\gamma + (1-\lambda)\gamma
\end{aligned} \quad (4)$$

From Eqs. (3) and (4), we get $a_2 + \lambda\omega(a_1, a_2) \in H_\gamma$ as required.

Proposition 2.9. Let A be an E-invex set with respect to $E \circ \omega$ and f is exponentially E-preinvex on A. If E is linear and idempotent. Then, $E - H_\gamma$ is an E-invex set with respect to $E \circ \omega$ for each $\gamma \in \mathbb{R}$.

Proof. Let $\gamma \in \mathbb{R}$ and $a_1, a_2 \in E - H_\gamma$. Then $e^{f(Ea_1)} \leq \gamma$ and $e^{f(Ea_2)} \leq \gamma$. Since A is an E-invex set with respect to $E \circ \omega$ then,

$$Ea_2 + \lambda(E \circ \omega)(Ea_1, Ea_2) \in A \quad (5)$$

and,

$$\begin{aligned}
&e^{f(E(Ea_2+\lambda(E\circ\omega)(Ea_1,Ea_2))} = e^{f\left(E^2a_2+\lambda\left(E^2\circ\omega\right)(Ea_1,Ea_2)\right)} \\
&= e^{f(Ea_2+\lambda(E\circ\omega)(Ea_1,Ea_2))}, \\
&\leq \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)} \leq \gamma
\end{aligned} \quad (6)$$

where the assumptions on E and f are employed in the last steps. Using Eqs. (5) and (6), the required result is obtained.

Proposition 2.10. If f is exponentially E-preinvex on A. Then, $H_{\gamma,E}$ is an invex set, for all $\gamma \in \mathbb{R}$.

Proof. Take $\gamma \in \mathbb{R}$ and $Ea_1, Ea_2 \in H_{\gamma,E}$ such that $e^{f(Ea_1)} \leq \gamma$ and $e^{f(Ea_2)} \leq \gamma$. From Remark 2.7, $E(A)$ is an invex set, then

$$Ea_2 + \lambda\omega(Ea_1, Ea_2) \in E(A) \subset A \quad (7)$$

From the assumption property on f, we have

$$\begin{aligned}
e^{f(Ea_2+\lambda\omega(Ea_1,Ea_2))} &\leq \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)} \\
&\leq \lambda\gamma + (1-\lambda)\gamma = \gamma
\end{aligned} \quad (8)$$

From Eqs. (7) and (8), we get $Ea_2 + \lambda\omega(Ea_1, Ea_2) \in H_{\gamma,E}$. Therefore, $H_{\gamma,E}$ is an invex set as required.

Next, we provide some conditions for f to be exponentially E-preinvex in terms of the epigraph sets of a function introduced in Definitions 1.8-1.9.

Proposition 2.11. If f is exponentially E-preinvex on A. Then $epi_E f$ is an invex set.

Proof. Let $(Ea_1, \alpha), (Ea_2, \beta) \in epi_E f$. From the definition of $epi_E f$, we have $e^{f(Ea_1)} \leq \alpha, e^{f(Ea_2)} \leq \beta$ and

$Ea_1, Ea_2 \in E(A)$. Since $E(A)$ is an invex set, then for any $\lambda \in [0,1]$

$$Ea_2 + \lambda\omega(Ea_1, Ea_2) \in E(A) \subseteq A \tag{9}$$

Since f is exponentially E-preinvex function, then

$$\begin{aligned} e^{f(Ea_2+\lambda\omega(Ea_1,Ea_2))} &\leq \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)} \\ &\leq \lambda\alpha + (1-\lambda)\beta \end{aligned} \tag{10}$$

From Eqs. (9) and (10), we get

$(Ea_2 + \lambda\omega(Ea_1, Ea_2), \lambda\alpha + (1-\lambda)\beta) \in epi_E f$.
i.e., $epi_E f$ is an invex set.

Proposition 2.12. Let $epi_E f$ is s. invex with respect to $E(A) \times \mathbb{R}$ then f is exponentially E-preinvex on A.

Proof. Let $a_1, a_2 \in A$ and $\lambda \in [0,1]$ such that

$$\left(Ea_1, e^{f(Ea_1)}\right), \left(Ea_2, e^{f(Ea_2)}\right) \in epi_E f \cap (E(A) \times \mathbb{R})$$

. From the invexity of $E(A)$,

$$Ea_2 + \lambda\omega(Ea_1, Ea_2) \in E(A),$$

thus,

$$\left(Ea_2 + \lambda\omega(Ea_1, Ea_2), \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)}\right) \in E(A) \times \mathbb{R}.$$

From the assumption on $epi_E f$

$$\left(Ea_2 + \lambda\omega(Ea_1, Ea_2), \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)}\right) \in epi_E f.$$

This means, $e^{f(Ea_2+\lambda\omega(Ea_1,Ea_2))} \leq \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)}$. Thus, f is exponentially E-preinvex on A.

Proposition 2.13. Let epif is s. invex with respect to $E(A) \times \mathbb{R}$ then f is exponentially E-preinvex on A.

Proof. Assume that $a_1, a_2 \in A$ and $\lambda \in [0,1]$ such that $\left(Ea_1, e^{f(Ea_1)}\right), \left(Ea_2, e^{f(Ea_2)}\right) \in \text{epif} \cap (E(A) \times \mathbb{R})$.
Since $E(A)$ is an invex set
Then $Ey + \lambda\omega(Ea_1, Ey) \in E(A)$, thus,

$$\left(Ea_2 + \lambda\omega(Ea_1, Ea_2), \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)}\right) \in E(A) \times \mathbb{R}.$$

Since epif is s. invex with respect to $E(A) \times \mathbb{R}$ then

$$\left(Ea_2 + \lambda\omega(Ea_1, Ea_2), \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)}\right) \in \text{epif}$$

This means, $e^{f(Ea_2+\lambda\omega(Ea_1,Ea_2))} \leq \lambda e^{f(Ea_1)} + (1-\lambda)e^{f(Ea_2)}$. Thus, f is exponentially E-preinvex on A.

3. Applications of exponentially E-preinvex functions to nonlinear optimization problems

Consider the following nonlinear optimization problem referred to as (P)

$$\begin{aligned} &\min f(a) \\ \text{s.t.}\quad &e^{g_i(a)} \leq c_i, \quad i = 1, \dots, r \\ &a \in A, \end{aligned}$$

where A, f and E are defined as in section one, $c_i \in \mathbb{R}$ $\forall i = 1, \dots, r$, and $g_i : \mathbb{R}^n \to \mathbb{R}$ be a real valued functions for $i = 1, \dots, r$ where A is E-invex set and f and g_i are exponentially E-preinvex functions on A, for each $i = 1, \dots, r$. The set of the feasible solution of problem (P) is referred to as F_e and defined as

$$F_e = \left\{a \in A : \quad e^{g_i(a)} \leq c_i, i = 1, \dots, r\right\}$$

Also, the set of optimal solutions denoted by $\text{argmin}_{F_e} f$ and defined as

$$\text{argmin}_{F_e} f = \{a^* \in F_e : f(a^*) \leq f(a) \quad \forall a \in F_e\}.$$

Problem (P) is named as exponentially E-preinvex optimization problem.

Using some assumptions, the sets F_e and $\text{argmin}_{F_e} f$ of problem (P) are s. E-invex with respect to $E(A)$ as it is stated in the below two theorems.

Theorem 3.1. Let $E(A)$ be an invex and $F_e \cap E(A) \neq \phi$. Then, the set F_e is s. E-invex with respect to $E(A)$.

Proof. Let $a_1^*, a_2^* \in F_e \cap E(A)$, i.e., $a_1^*, a_2^* \in E(A)$, then there exists $a_1, a_2 \in A$ such that $a_1^* = Ea_1, a_2^* = E, a_2$ and $a_2^* + \lambda\omega\left(a_1^*, a_2^*\right) \in E(A)$, for each $\lambda \in [0,1]$. From Proposition 1.4, $E(A) \subseteq A$, hence, $a_2^* + \lambda\omega\left(a_1^*, a_2^*\right) \in A$. We need to show that $a_2^* + \lambda\omega\left(a_1^*, a_2^*\right) \in F_e$. Fix $i \in \{1, 2, \dots, r\}$. Since g_i is exponentially E-preinvex, then $e^{g_i(a_2^*+\lambda\omega(a_1^*,a_2^*))} \leq \lambda e^{g_i(a_1^*)} + (1-\lambda)e^{g_i(a_2^*)} \leq c_i$. From the last expression and definition of F_e, one obtain $a_2^* + \lambda\omega\left(a_1^*, a_2^*\right) \in F_e$ as required.

Theorem 3.2. Let $E(A)$ be an invex set and $\text{argmin}_{F_e} f \cap E(A) \neq \phi$.Then, $\text{argmin}_{F_e} f$ is a s. E-invex with respect to $E(A)$.

Proof. Let $a_1^*, a_2^* \in \text{argmin}_{F_e} f \cap E(A)$ then $f\left(a_1^*\right) = f\left(a_2^*\right) = p^*$. Using Theorem 3.1, we have $a_2^* + \lambda\omega\left(a_1^*, a_2^*\right) \in E(A) \subseteq A$. Using the exponentially E-preinvexity of f,

$$\begin{aligned} e^{f(a_2^*+\lambda\omega(a_1^*,a_2^*))} &\leq \lambda e^{f(a_1^*)} + (1-\lambda)e^{f(a_2^*)} \\ &\leq \lambda e^{p^*} + (1-\lambda)e^{p^*} = e^{p^*}. \end{aligned}$$

i.e.,

$$f\left(a_2^* + \lambda\omega\left(a_1^*, a_2^*\right)\right) = p^*.$$

Thus,

$$a_2^* + \lambda\omega\left(a_1^*, a_2^*\right) \in \operatorname{argmin}_{F_e} f.$$

Under certain condition every local minimum of problem (P) is a global minimum as it is stated next.

Theorem 3.3. If $F_e \subset E(A)$. Then every local minimum is a global minimum.

Proof. Let $a^* \in F_e \subset E(A)$ be a local minimum point then there exists $\varepsilon > 0$ such that

$$\begin{aligned} B\left(a^*, \varepsilon\right) \subset E(A) \text{ and } f\left(a^*\right) \leq f(a) \\ \forall a \in U = B\left(a^*, \varepsilon\right) \cap F_e. \end{aligned} \tag{11}$$

To complete the proof, it is enough to show that $f\left(a^*\right) \leq f(a), \quad \forall a \in F_e \backslash U$. On contrary, assume that there is $\bar{a} \in F_e, \bar{a} \neq a^*$ such that

$$f(\bar{a}) < f\left(a^*\right). \tag{12}$$

From Eq. (11), $\bar{a} \notin B\left(a^*, \varepsilon\right)$ and $\|\bar{a} - a^*\| \geq \varepsilon$. Let $a_1^*, a_2^* \in A$ such that $\bar{a} = Ea_1^*, a^* = Ea_2^*$. Since A is E-invex, we have $Ea_2^* + \lambda\omega\left(Ea_1^*, Ea_2^*\right) \in A$. Since f is exponentially E-preinvex on A, then

$$e^{f\left(a_2^* + \lambda\omega\left(a_1^*, a_2^*\right)\right)} \leq \lambda e^{f\left(a_1^*\right)} + (1-\lambda) e^{f\left(a_2^*\right)}$$

Applying Eq. (12), the last inequality gives

$$\begin{aligned} e^{f\left(a^* + \lambda\omega\left(\bar{a}, a^*\right)\right)} &\leq \lambda e^{f(\bar{a})} + (1-\lambda) e^{f\left(a^*\right)} \\ &< \lambda f\left(a^*\right) + (1-\lambda) e^{f\left(a^*\right)} = e^{f\left(a^*\right)} \end{aligned} \tag{13}$$

If $\omega\left(\bar{a}, a^*\right) = 0$. Then for any $\lambda \in [0,1]$, it yields $f\left(a^* + \lambda\omega\left(\bar{a}, a^*\right)\right) = f\left(a^*\right)$ which contradicts Eq. (13). If $\omega\left(\bar{a}, a^*\right) \neq 0$. Choose $\varepsilon > 0$ sufficiently small such that $\frac{\varepsilon}{\|\omega(\bar{a}, a^*)\|} \leq 1$. Set $\bar{\lambda} = \min\left\{\lambda_{a_1^*, a_2^*}, \frac{\varepsilon}{\|\omega(\bar{a}, a^*)\|}\right\}$. Then for any $\lambda \in [0, \bar{\lambda}]$, we get

$$\left\|a^* - \left[a^* + \lambda\omega\left(\bar{a}, a^*\right)\right]\right\| = \left\|\lambda\omega\left(\bar{a}, a^*\right)\right\| \leq \bar{\lambda}\left\|\omega\left(\bar{a}, a^*\right)\right\| \leq \varepsilon,$$

i.e., $a^* + \lambda\omega\left(\bar{a}, a^*\right) \in B\left(a^*, \varepsilon\right) \subset E(A)$. From the last statement and the assumption $F_e \subset E(A)$, then employing Theorem 3.1, we have $a^* + \lambda\omega\left(\bar{a}, a^*\right) \in F_e$. Again, Eq. (13) contradicts the fact that a^* is a local minimum on F_e.

References

[1] R. T. Rockafellar. *Convex analysis*. ***18***. Princeton university press, 1970.

[2] I. Ekeland and R. Temam. *Convex analysis and variational problems*. Society for Industrial and Applied Mathematics, 1999.

[3] C. Zalinescu. *Convex analysis in general vector spaces*. World scientific, 2002.

[4] S. N. Majeed, (2023) *"Strong Fenchel Duality For Evenly Convex Optimization Problems"* **Journal of Applied Science and Engineering** ***26***(12): 1723–1729. DOI: 10.6180/jase.202312_26(12).0004.

[5] C. Niculescu and L.-E. Persson. *Convex functions and their applications*. ***23***. Springer, 2006.

[6] G. Cristescu and L. Lupsa. *Non-connected convexities and applications*. ***68***. Springer Science & Business Media, 2013.

[7] M. A. Hanson, (1981) *"On sufficiency of the Kuhn-Tucker conditions"* **Journal of Mathematical Analysis and Applications** ***80***(2): 545–550. DOI: 10.1016/0022-247X(81)90123-2.

[8] R. Nehse, (1979) *"Problems in connection with the Hahn-Banach-Theorem"* **Proc. of the Summer-school" Nonlinear Analysis", Berlin**:

[9] B. Craven, (1981) *"Invex functions and constrained local minima"* **Bulletin of the Australian Mathematical society** ***24***(3): 357–366. DOI: 10.1017/S0004972700004895.

[10] B. Craven. *Duality for generalized convex fractional programs*. Academic Press New York, 1981, 473–489.

[11] A. Ben-Israel and B. Mond, (1986) *"What is invexity?"* **The ANZIAM Journal** ***28***(1): 1–9. DOI: 10.1017/S0334270000005142.

[12] M. Hanson and B. Mond, (1987) *"Convex transformable programming problems and invexity"* **Journal of Information and Optimization Sciences** ***8***(2): 201–207. DOI: 10.1080/02522667.1987.10698886.

[13] T. Weir and B. Mond, (1988) *"Pre-invex functions in multiple objective optimization"* **Journal of Mathematical Analysis and applications** ***136***(1): 29–38. DOI: 10.1016/0022-247X(88)90113-8.

[14] S. Mohan and S. Neogy, (1995) *"On invex sets and preinvex functions"* **Journal of Mathematical Analysis and Applications** ***189***(3): 901–908. DOI: 10.1006/jmaa.1995.1057.

[15] M. A. Noor and K. I. Noor, (2006) *"Some characterizations of strongly preinvex functions"* **Journal of Mathematical Analysis and Applications** ***316***(2): 697–706. DOI: 10.1016/j.jmaa.2005.05.014.

[16] M. A. Noor, (2007) *"On Hadamard integral inequalities involving two log-preinvex functions"* **J. Inequal. Pure Appl. Math** ***8***(3): 1–14.

[17] S.-l. Chen, N.-J. Huang, D. O'Regan, et al., (2014) *"Geodesic B-preinvex functions and multiobjective optimization problems on Riemannian manifolds"* **Journal of Applied Mathematics *2014***: DOI: 10.1155/2014/524698.

[18] S. N. Majeed, (2019) *"Fuzzy preinvexity via ranking value functions with applications to fuzzy optimization problems"* **Journal of Interdisciplinary Mathematics 22**(8): 1485–1494. DOI: 10.1080/09720502.2019.1706846.

[19] M. B. Khan, M. A. Noor, L. Abdullah, and Y.-M. Chu, (2021) *"Some new classes of preinvex fuzzy-interval-valued functions and inequalities"* **International Journal of Computational Intelligence Systems *14***(1): 1403–1418. DOI: 10.2991/ijcis.d.210409.001.

[20] M. Khan, G. Santos-García, M. Noor, and M. Soliman, (2022) *"New class of preinvex fuzzy mappings and related inequalities"* **Mathematics *10***: 3753. DOI: 10.3390/math10203753.

[21] A. A. Shaikh, A. Iqbal, and C. K. Mondal, (2018) *"Some results on φ-convex functions and geodesic φ-convex functions"* **Differential Geometry- Dynamical System *20***: 159–169.

[22] A. A. Shaikh, C. K. Mondal, and I. Ahmad, (2019) *"Non-existence of certain type of convex functions on a Riemannian manifold with a pole"* **Journal of Geometry and Physics *140***: 104–110. DOI: 10.1016/j.geomphys.2019.02.006.

[23] A. A. Shaikh, R. P. Agarwal, and C. K. Mondal, (2020) *"Geodesic sandwich theorem with an application"* **Mathematical Inequalities and Applications 23**: 161–167. DOI: 10.7153/mia-2020-23-13.

[24] E. Youness, (1999) *"E-convex sets, E-convex functions, and E-convex programming"* **Journal of Optimization Theory and Applications *102***: 439–450. DOI: org/10.1023/A:1021792726715.

[25] E. A. Youness et al., (2001) *"Stability in E-convex programming"* **International Journal of Mathematics and Mathematical Sciences *26***: 643–648. DOI: 10.1155/S0161171201006317.

[26] E. A. Youness, (2001) *"Optimality criteria in E-convex programming"* **Chaos, Solitons & Fractals 12**(9): 1737–1745. DOI: 10.1016/S0960-0779(00)00036-9.

[27] A. Megahed, H. Gomma, E. Youness, and A. El-Banna, (2013) *"A study on the duality of E-convex non-linear programming problem"* **International Journal of Mathematical Analysis 7**(4): 175–185. DOI: 10.12988/ijma.2013.13015.

[28] A. E.-M. A. Megahed, H. G. Gomma, E. A. Youness, and A.-Z. H. El-Banna, (2013) *"Optimality conditions of E-convex programming for an E-differentiable function"* **Journal of Inequalities and Applications *2013***: 1–11. DOI: 10.1186/1029-242X-2013-246.

[29] X. Chen, (2002) *"Some properties of semi-E-convex functions"* **Journal of Mathematical Analysis and Applications *275***(1): 251–262. DOI: 10.1016/S0022-247X(02)00325-6.

[30] M. Soleimani-Damaneh, (2011) *"E-convexity and its generalizations"* **International Journal of Computer Mathematics *88***(16): 3335–3349. DOI: 10.1080/00207160.2011.589899.

[31] N. Abdulaleem. "E-invexity and generalized E-invexity in E-differentiable multiobjective programming". In: *ITM Web of Conferences*. ***24***. EDP Sciences. 2019, 01002. DOI: 10.1051/itmconf/20192401002.

[32] S. Majeed and A. Enad. "On semi strongly (E, F)-convex functions and semi strongly (E, F)-convex optimization problems". In: *Journal of Physics: Conference Series*. ***1879***. 2. IOP Publishing. 2021, 022110. DOI: 10.1088/1742-6596/1879/2/022110.

[33] W. Saleh, (2022) *"Hermite–Hadamard type inequality for (E, F)-convex functions and geodesic (E, F)-convex functions"* **RAIRO-Operations Research *56***(6): 4181–4189. DOI: 10.1051/ro/2022185.

[34] M. Elbrolosy, (2022) *"Semi-(E, F)-convexity in complex programming problems"* **AIMS Mathematics 7**(6): 11119–11131. DOI: 10.3934/math.2022621.

[35] R. E. Hazim and S. N. Majeed, (2023) *"Some properties of semi (p, e)-convexity"* **Journal of Interdisciplinary Mathematics *26***(4): 629–635. DOI: 10.47974/JIM-1481.

[36] N. Abdulaleem, (2023) *"Optimality and duality for E-differentiable multiobjective programming problems involving E-type functions"* **Journal of Industrial and Management Optimization *19***(2): 1513–1527. DOI: 10.3934/jimo.2022004.

[37] N. Abdulaleem, (2023) *"E-univex sets, E-univex functions and E-differentiable E-univex programming"* **Results in Mathematics *78***(1): 3. DOI: 10.1007/s00025-022-01775-5.

[38] N. Abdulaleem, (2023) *"Optimality conditions for a class of E-differentiable vector optimization problems with interval-valued objective functions under E-invexity"* **International Journal of Computer Mathematics *100***(7): 1601–1624. DOI: 10.1080/00207160.2023.2207389.

[39] C. Fulga and V. Preda, (2009) *"Nonlinear programming with E-preinvex and local E-preinvex functions"* **European Journal of Operational Research *192***(3): 737–743. DOI: 10.1016/j.ejor.2007.11.056.

[40] Z. Luo et al., (2011) *"Some properties of semi-E-preinvex maps in Banach spaces"* **Nonlinear Analysis: Real World Applications *12***(2): 1243–1249. DOI: 10.1016/j.nonrwa.2010.09.019.

[41] S. N. Majeed, (2019) *"Strongly and Semi Strongly E_h-b-Vex Functions: Applications to Optimization Problems"* **Iraqi Journal of Science**: 2022–2029. DOI: 10.24996/ijs.2019.60.9.16.

[42] N. Abdulaleem, (2021) *"EB-invexity in E-differentiable mathematical programming"* **Results in Control and Optimization *4***: 100046. DOI: 10.1016/j.rico.2021.100046.

[43] A. A. Shaikh, C. K. Mondal, and R. P. Agarwal, (2021) *"Some properties of geodesic (α, E)-preinvex functions on Riemannian manifolds"* **Khayyam Journal of Mathematics *7***(2): 201–210. DOI: 10.22034/KJM.2020.220226.1717.

[44] A. Iqbal and P. Kumar, (2023) *"Geodesic $\mathcal{E}$-prequasi-invex function and its applications to non-linear programming problems"* **Numerical Algebra, Control and Optimization *13***(1): 1–10. DOI: 10.3934/naco.2021040.

[45] G. Alirezaei and R. Mathar. "On exponentially concave functions and their impact in information theory". In: *2018 Information Theory and Applications Workshop (ITA)*. IEEE. 2018, 1–10. DOI: 10.1109/ITA.2018.8503202.

[46] S. Pal and T.-K. L. Wong, (2018) *"Exponentially concave functions and a new information geometry"* **The Annals of probability *46***(2): 1070–1113.

[47] M. A. Noor and K. I. Noor, (2019) *"On exponentially convex functions"* **JOURNAL OF ORISSA MATHEMATICAL SOCIETY *975***: 2323.

[48] M. Noor and K. Noor, (2019) *"Strongly exponentially convex functions and their properties"* **Journal of Advanced Mathematical Studies *12***(2): 177–185.

[49] M. Noor and K. Noor, (2019) *"Some properties of differentiable exponential convex functions"* **preprint, Math, Deparment, COMSATS university, Islamabad, Pakistan**:

[50] M. A. Noor and K. I. Noor, (2019) *"New classes of strongly exponentially preinvex functions"* **AIMS Math *4***(6): 1554–1568. DOI: 10.3934/math.2019.6.1554.

[51] M. A. Noor, K. I. Noor, and T. M. Rassias, (2021) *"Relative strongly exponentially convex functions"* **Nonlinear Analysis and Global Optimization**: 357–371. DOI: 10.1007/97.

[52] J. Bisht, N. Sharma, S. K. Mishra, and A. Hamdi, (2023) *"Some new integral inequalities for higher-order strongly exponentially convex functions"* **Journal of Inequalities and Applications *2023***(1): 41. DOI: 10.1186/s13660-023-02952-y.

[53] M. A. Noor and K. I. Noor, (2019) *"Some properties of exponentially preinvex functions"* **Facta Universitatis, Series: Mathematics and Informatics**: 941–955. DOI: 10.22190/FUMI1905941N.

[54] N. Abdulaleem, (2024) *"Exponentially E-convex vector optimization problems"* **Journal of Industrial and Management Optimization**: 0–0. DOI: 10.3934/jimo.2023164.