Free Vibration Behaviour of Functionally Graded Plates Using Higher-Order Shear Deformation Theory

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Abstract

The prime aim of the present study is to develop analytical formulations and solutions for the free vibration analysis of functionally graded plates (FGPs) using higher order shear deformation theory (HSDT) without enforcing zero transverse shear stress on the top and bottom surfaces of the plate. The theoretical model presented herein incorporates the transverse extensibility which accounts for the transverse effects. The equations of equilibrium and boundary conditions are derived using the principle of virtual work. Solutions are obtained for FGPs in closed-form using Navier's technique and solving the eigen value equation. The present results are compared with the solutions of the other HSDTs available in the literature. It can be concluded that the proposed theory is accurate and efficient in predicting the vibration behaivour of functionally graded plates.

Key Words: Vibration Analysis, Functionally Graded Plates, HSDT, Navier's Method

1. Introduction

Functionally graded materials (FGMs) are a new generation of engineered materials in which the material properties are continually varied through the thickness direction by mixing two different materials and thus no distinct internal boundaries exist and failures from interfacial stress concentrations developed in conventional structural components can be avoided. FGMs are widely used in many structural applications such as mechanics, civil engineering, optical, electronic, chemical, mechanical, biomedical, energy sources, nuclear, automotive fields and ship building industries to eliminate stress concentration and relax residual stresses and enhance bond strength [1]. Most structures, irrespective of their use, will be subjected to dynamic loads during their operational life. Increased use of FGMs in various structural applications necessitates the development of accurate theoretical models to predict their response.

The literature on the FGPs is relatively scarce when compared to isotropic and laminated plates. Because of FGMs applications in high temperature environments, most of the studies on the behavior of FGPs focus on the thermo-mechanical response of FGPs: Reddy and Chin [2], Reddy [3], Vel and Batra [4,5], Cheng and Batra [6] and Javaheri and Eslami [7].

In the past, a variety of plate theories have been proposed to study vibration behavior of FGPs. The classical plate theory (CPT) neglects the transverse shear effects and provides acceptable results of thin plates only. However, for moderately thick plates CPT under estimates deflections and over estimates buckling loads and natural frequencies. The first-order shear deformation theories (FSDTs) are based on Reissner [8] and Mindlin [9] accounts for the transverse shear deformation effect by means of a linear variation of in-plane displacements and

stresses through the thickness of the plate, but requires a correction factor to satisfy the free transverse shear stress conditions on the top and bottom surfaces of the plate. Although, the FSDT provides a faithful description of the mechanics for thin to moderately thick plates, it is not convenient to use due to difficulty with determination of the correct value of shear correction factor [10]. In-order to overcome the limitations of FSDT many HSDTs were developed that involve higher order terms in Taylors-series expansion of the displacements in the thickness coordinate, notable among them are Reddy [3], Zenkour [11–13], Kant and Co-workers [14–19], Kadkhodayan [20], Matsunaga [21,22], Xiang [23] and Ferreira et al. [24]. Sahmani and Ansari [25] investigated the free vibration behavior of FGM micro plates using strain gradient elasticity and higher-order shear deformable plate theory. They modeled the FGM microplates using simple power law function and Mori-Tanaka homogenization technique. Saidi et al. [26,27] presented a new formulation of the Navier equations of motion for solving the known three-dimensional elastostatics and elastodynamics problems and an exact analytical solution for free vibration of thin rectangular FGPs based on classical plate theory. They also studied the effects of in-plane displacement on the vibration of rectangular FGPs. Maziar and Iman [28] used the finite element method to study the vibration behavior of functionally graded plates with multiple circular and noncircular cutouts. Vibration problems of FGPs can be found in Batra and Jin [29], Ferreira et al. [30], Vel and Batra [31] presented the valid exact solutions for thick and thin plates, and for arbitrary variation of material properties in the thickness direction, Reddy and Phan [32], Roque et al. [33], Cheng and Batra [34], Mallikarjuna and Kant [35], Zhao et al. [36], Hosseini-Hashemi [37], Pradyumna and Bandyopadhyay [38], Fares et al. [39], Mohammad Talha and Singh [40], Hassen et al. [41], Hosseini-Hashemi et al. [42], Putcha and Reddy [43] and Marur and Kant [44], Liu et al. [45], Mirtalaie et al. [46], Sina et al. [47] and Fallah et al. [48]. Most of these theories do not account for transverse shear stress on the top and bottom surfaces of the plate and transverse extensibility by neglecting the transverse stress in the z-direction (σ_z). Neves et al. [49,50] derived a higher order shear deformation theory (HSDT) for modeling of functionally graded material plates and focused on the thickness stretching issue on the static, free vibration, and buckling analysis of FGPs by a meshless technique. They used the virtual work principle of displacements under Carrera's Unified Formulation (CUF) to obtain the governing equations and boundary conditions [49]. Qian et al. [51] analyzed the static Static deformations, and free and forced vibrations of a thick rectangular functionally graded elastic plate are analyzed by using a higher-order shear and normal deformable plate theory (HOSNDPT) and a meshless local Petrov-Galerkin (MLPG) method. Hence, Vel and Batra [31] and Qian et al. [51] are considered as reference to validate the present results in this paper.

The present paper deals with the analytical formulations and solutions for the vibration analysis of functionally graded plates (FGPs) using higher order shear deformation theory (HSDT) without enforcing zero transverse shear stress on the top and bottom surfaces of the plate. The theoretical model presented herein incorporates the transverse extensibility which accounts for the transverse effects. Thus a shear correction factor is not required. The plate material is graded through the thickness direction. The plate's governing equations and its boundary conditions are derived by employing the principle of virtual work. Solutions are obtained for FGPs in closed-form using Navier's technique and solving the eigen value equation. The present results are compared with the solutions of the other HSDTs available in the literature to verify the accuracy of the proposed theory in predicting the natural frequencies of FGPs. The effect of side-to-thickness ratios, aspect ratios and modulus ratios and volume fraction exponent on the natural frequencies are studied after establishing the accuracy of the present results for FGPs.

2. Theoretical Formulation

In formulating the higher-order shear deformation theory, a rectangular plate of length *a*, width *b* and thickness *h* is consider, that composed of functionally graded material through the thickness. Figure 1 shows the functionally graded material plate with the rectangular Cartesian coordinate system x, y and z. The material properties are assumed to be varied in the thickness direction only and the bright and dark areas correspond to ceramic
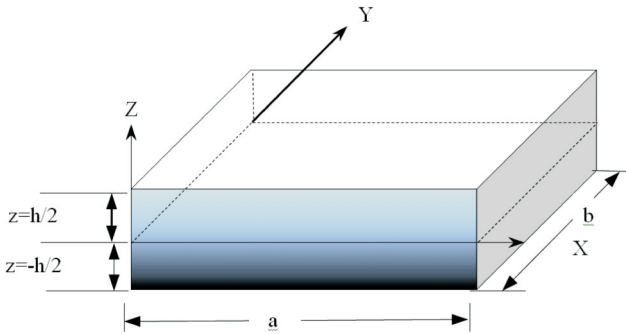
Figure 1. Functionally graded plate and coordinates.

and metal particles respectively. On the top surface ($z =$ +$h/2$), the plate is composed of entirely ceramic and graded to the bottom surface ($z = -h/2$) that composed of entirely metallic. The reference surface is the middle surface of the plate ($z = 0$). The functionally graded material plate properties are assumed to be the function of the volume fraction of constituent materials. The functional relationship between the material property and the thickness coordinate is assumed to be

$$P(z) = (P_t - P_b)\left(\frac{z}{h} + \frac{1}{2}\right)^n + P_b \qquad (1)$$

where P denotes the effective material property, P_t, and P_b denotes the property on the top and bottom surface of the plate respectively and n is the material variation parameter that dictates the material variation profile through the thickness. The effective material properties of the plate, including Young's modulus, E, density ρ, and shear modulus, G, vary according to Eq. (1), and poisons ratio (υ) is assumed to be constant.

2.1 Displacement Models

In order to approximate 3D plate problem to a 2D one, the displacement components $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$ at any point in the plate are expanded in terms of the thickness coordinate. The elasticity solution indicates that the transverse shear stress varies parabolically through the plate thickness. This requires the use of a displacement field, in which the in-plane displacements are expanded as cubic functions of the thickness coordinate. In addition, the transverse normal strain may vary nonlinearly through the plate thickness. The displacement field which satisfies the above criteria may be assumed in the form [52]:

$$\begin{aligned}
u(x, y, z) &= u_o(x, y) + z\theta_x(x, y) + z^2 u_o^*(x, y) + z^3\theta_x^*(x, y) \\
v(x, y, z) &= v_o(x, y) + z\theta_y(x, y) + z^2 v_o^*(x, y) + z^3\theta_y^*(x, y) \\
w(x, y, z) &= w_o(x, y) + z\theta_z(x, y) + z^2 w_o^*(x, y) + z^3\theta_z^*(x, y)
\end{aligned} \tag{2}$$

where u_0, v_0 is the in-plane displacements of a point (x, y) on the mid plane. w_o is the transverse displacement of a point (x, y) on the mid plane. θ_x, θ_y, θ_z are rotations of the normal to the mid plane about y and x-axes. u_0^*, v_0^*, w_0^*, θ_x^*, θ_y^*, and θ_z^* are the corresponding higher order deformation terms.

By substitution of displacement relations from Eq. (2) into the strain displacement equations of the classical theory of elasticity the following relations are obtained:

$$\begin{aligned}
\varepsilon_x &= \varepsilon_{xo} + zk_x + z^2\varepsilon_{xo}^* + z^3 k_x^* \\
\varepsilon_y &= \varepsilon_{yo} + zk_y + z^2\varepsilon_{yo}^* + z^3 k_y^* \\
\varepsilon_z &= \varepsilon_{zo} + zk_z + z^2\varepsilon_{zo}^* \\
\gamma_{xy} &= \varepsilon_{xyo} + zk_{xy} + z^2\varepsilon_{xyo}^* + z^3 k_{xy}^* \\
\gamma_{yz} &= \phi_y + zK_{yz} + z^2\phi_y^* + z^3 k_{yz}^* \\
\gamma_{xz} &= \phi_x + zK_{xz} + z^2\phi_x^* + z^3 k_{xz}^*
\end{aligned} \tag{3}$$

2.2 Elastic Stress-Strain Relations

The elastic stress-strain relations depend on which assumption of ε_z we consider. If $\varepsilon_z \neq 0$, i.e., thickness stretching is allowed then the 3D model is used. In the case of functionally graded materials the constitutive equations can be written as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{4}$$

where (σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{xz}) are the stresses and (ε_x, ε_y, ε_z, γ_{xy}, γ_{yz}, γ_{xz}) are the strains with respect to the axes, Q_{ij}'s are the plane stress reduced elastic coefficients in the plate axes that vary through the plate thickness given by

$$Q_{11} = Q_{22} = Q_{33} = \frac{(1-\upsilon^2)E(z)}{1-3\upsilon^2-2\upsilon^3}; \quad Q_{12} = Q_{13} = Q_{23} = \frac{\upsilon(1+\upsilon)E(z)}{1-3\upsilon^2-2\upsilon^3};$$

$$Q_{44} = Q_{55} = Q_{66} = \frac{E(z)}{2(1+\upsilon)}; \quad E(z) = (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^n + E_m \tag{5}$$

where E_c is the modulus of Elasticity of the ceramic material and E_m is the modulus of elasticity of the metal.

2.3 Governing Equations of Motion

The governing equations of motion of present theory are derived using the Hamilton's principle can be written in the analytical form as:

$$\int_0^T (\delta U + \delta V - \delta K)dt = 0 \tag{6}$$

where δU is the virtual strain energy, δV is the virtual work done by applied forces, and δK is the virtual kinetic energy and is given by:

$$\delta U = \int_A \left\{ \int_{-h/2}^{h/2} \left[\sigma_x \delta \in_x + \sigma_y \delta \in_y + \sigma_z \delta \in_z \right. \right.$$
$$\left. \left. + \tau_{xy} \delta \gamma_{xy} + \tau_{xz} \delta \gamma_{xz} + \tau_{yz} \delta \gamma_{yz} \right] dz \right\} dxdy \tag{7}$$

$$\delta V = -\int q \delta w^+ dxdy \tag{8}$$

where $w^+ = w_0 + \dfrac{h}{2}\theta_z + \dfrac{h^2}{4}w_0^* + \dfrac{h^3}{8}\theta_z^*$ is the transverse displacement of any point on the top surface of the plate and q is the transverse load applied at the top surface of the plate.

$$\delta K = \int_A \left\{ \int_{-h/2}^{h/2} \rho_0 \left[(\dot{u}_0 + Z\dot{\theta}_x + Z^2 \dot{u}_0^* + Z^3 \dot{\theta}_x^*)(\delta \dot{u}_0 + Z\delta\dot{\theta}_x \right. \right.$$
$$+ Z^2 \delta \dot{u}_0^* + Z^3 \delta\dot{\theta}_x^*) + (\dot{v}_0 + Z\dot{\theta}_y + Z^2 \dot{v}_0^* + Z^3 \dot{\theta}_y^*)(\delta \dot{v}_0 + Z\delta\dot{\theta}_y$$
$$+ Z^2 \delta \dot{v}_0^* + Z^3 \delta\dot{\theta}_y^*) + (\dot{w}_0 + z\dot{\theta}_z + z^2 \dot{w}_0^* + z^3 z\dot{\theta}_0^*)(\delta \dot{w}_0 + z\delta\dot{\theta}_z$$
$$\left. \left. + z^2 \delta\dot{w}_0^* + z^3 \delta z\dot{\theta}_0^*) \right] dz \right\} dxdy \tag{9}$$

Substituting for δU, δV and δK in the virtual work statement in Eq. (6) and integrating through the thickness, integrating by parts and collecting the coefficients of δu_o, δv_o, δw_o, $\delta\theta_x$, $\delta\theta_y$, $\delta\theta_z$, δu_o^*, δv_o^*, δw_o^*, $\delta\theta_x^*$, $\delta\theta_y^*$, $\delta\theta_z^*$ the following equations of motion are obtained.

$$\delta u_0 : \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_1 \ddot{u}_0 + I_2 \ddot{\theta}_x + I_3 \ddot{u}_0^* + I_4 \ddot{\theta}_x^*$$

$$\delta v_0 : \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = I_1 \ddot{v}_0 + I_2 \ddot{\theta}_y + I_3 \ddot{v}_0^* + I_4 \ddot{\theta}_y^*$$

$$\delta w_0 : \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = I_1 \ddot{w}_0 + I_2 \ddot{\theta}_z + I_3 \ddot{w}_0^* + I_4 \ddot{\theta}_z^*$$

$$\delta\theta_x : \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2 \ddot{u}_0 + I_3 \ddot{\theta}_x + I_4 \ddot{u}_0^* + I_5 \ddot{\theta}_x^*$$

$$\delta\theta_y : \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = I_2 \ddot{v}_0 + I_3 \ddot{\theta}_y + I_4 \ddot{v}_0^* + I_5 \ddot{\theta}_y^*$$

$$\delta\theta_z : \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} - N_z + \frac{h}{2}(q) = I_2 \ddot{w}_0 + I_3 \ddot{\theta}_z + I_4 \ddot{w}_0^* + I_5 \ddot{\theta}_z^*$$

$$\delta u_0^* : \frac{\partial N_x^*}{\partial x} + \frac{\partial N_{xy}^*}{\partial y} - 2S_x = I_3 \ddot{u}_0 + I_4 \ddot{\theta}_x + I_5 \ddot{u}_0^* + I_6 \ddot{\theta}_x^*$$

$$\delta v_0^* : \frac{\partial N_y^*}{\partial y} + \frac{\partial N_{xy}^*}{\partial x} - 2S_y = I_3 \ddot{v}_0 + I_4 \ddot{\theta}_y + I_5 \ddot{v}_0^* + I_6 \ddot{\theta}_y^*$$

$$\delta w_0^* : \frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} - 2M_z + \frac{h^2}{4}(q) = I_3 \ddot{w}_0 + I_4 \ddot{\theta}_z + I_5 \ddot{w}_0^* + I_6 \ddot{\theta}_z^*$$

$$\delta\theta_x^* : \frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - 3Q_x^* = I_4 \ddot{u}_0 + I_5 \ddot{\theta}_x + I_6 \ddot{u}_0^* + I_7 \ddot{\theta}_x^*$$

$$\delta\theta_y^* : \frac{\partial M_y^*}{\partial y} + \frac{\partial M_{xy}^*}{\partial x} - 3Q_y^* = I_4 \ddot{v}_0 + I_5 \ddot{\theta}_y + I_6 \ddot{v}_0^* + I_7 \ddot{\theta}_y^*$$

$$\delta\theta z^* : \frac{\partial S_x^*}{\partial x} + \frac{\partial S_y^*}{\partial y} - 3N_z^* + \frac{h^3}{8}(q) = I_4 \ddot{w}_0 + I_5 \ddot{\theta}_z + I_6 \ddot{w}_0^* + I_7 \ddot{\theta}_z^*$$

$$\tag{10}$$

where the in-plane force and moment resultants are defined as:

$$\begin{Bmatrix} N_x & | & N_x^* \\ N_y & | & N_y^* \\ N_z & | & N_z^* \\ N_{xy} & | & N_{xy}^* \end{Bmatrix} = \sum_{L=1}^{n} \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \end{Bmatrix} \left[1 \,|\, z^2 \right] dz \tag{11}$$

$$\begin{Bmatrix} M_x & | & M_x^* \\ M_y & | & M_y^* \\ M_z & | & 0 \\ M_{xy} & | & M_{xy}^* \end{Bmatrix} = \sum_{L=1}^{n} \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \end{Bmatrix} \left[Z \,|\, Z^3 \right] dz \tag{12}$$

and the transverse force resultants and inertias are given by:

$$\begin{Bmatrix} Q_x & | & Q_x^* \\ Q_y & | & Q_y^* \end{Bmatrix} = \sum_{L=1}^{n} \int_{-h/2}^{h/2} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} \left[1 \,|\, Z^2 \right] dz \tag{13}$$

$$\begin{Bmatrix} S_x & | & S_x^* \\ S_y & | & S_y^* \end{Bmatrix} = \sum_{L=1}^{n} \int_{-h/2}^{h/2} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} \left[Z \,|\, Z^3 \right] dz \tag{14}$$

$$I_1, I_2, I_3, I_4, I_5, I_6, I_7 = \int_{-h/2}^{h/2} \rho_0 (1, Z, Z^2, Z^3, Z^4, Z^5, Z^6) dz \quad (15)$$

where I_1, I_2, I_3, I_4, I_5, I_6, and I_7 are the mass moments of inertia. The terms involving I_2, I_3, I_4, I_5, I_6, and I_7 are called rotary inertia terms. The terms can contribute to higher-order vibration or frequency modes.

The resultants in Equations (11)–(14) can be related to the total strains in Eq. (4) by the following matrix:

$$\begin{Bmatrix} N \\ N^* \\ -- \\ M \\ M^* \\ -- \\ Q \\ Q^* \end{Bmatrix} = \begin{bmatrix} A & | & B & | & 0 \\ \overline{B^t} & | & \overline{D_b} & | & \overline{0} \\ \overline{0} & | & \overline{0} & | & \overline{D_s} \end{bmatrix} \begin{Bmatrix} \in_0 \\ \in_0^* \\ -- \\ K_s \\ K^* \\ -- \\ \phi \\ \phi^* \end{Bmatrix} \quad (16)$$

where $N = [N_x\ N_y\ N_z\ N_{xy}]^t$; $N^* = [N_x^*\ N_y^*\ N_z^*\ N_{xy}^*]^t$; $M = [M_x\ M_y\ M_z\ M_{xy}]^t$; $M^* = [M_x^*\ M_y^*\ M_z^*\ M_{xy}^*]^t$; $Q = [Q_x\ Q_y\ S_x\ S_y]^t$; $Q^* = [Q_x^*\ Q_y^*\ S_x^*\ S_y^*]^t$; $\varepsilon_0 = [\varepsilon_{x0}\ \varepsilon_{y0}\ \varepsilon_{z0}\ \varepsilon_{xy0}]^t$; $\varepsilon_0^* = [\varepsilon_{x0}^*\ \varepsilon_{y0}^*\ \varepsilon_{z0}^*\ \varepsilon_{xy0}^*]^t$; $K = [K_x\ K_y\ K_z\ K_{xy}]^t$; $K^* = [K_x^*\ K_y^*\ 0\ K_{xy}^*]^t$; $\phi = [\phi_x\ \phi_y\ k_{xz}\ k_{yz}]^t$; $\phi^* = [\phi_x^*\ \phi_y^*\ k_{xz}^*\ k_{yz}^*]^t$. The matrices [A], [B], [D] and [Ds] are the plate stiffness whose elements can be calculated using Eq. (4), and Eq. (11)–(14).

3. Analytical Solutions

Rectangular plates are generally classified by referring to the type of support used. We are here concerned with the analytical solutions of the Eq. (10)–(16) for simply supported FGPs. Exact solutions of the partial differential Eq. (10) an arbitrary domain and for general boundary conditions are difficult. Although, the Navier-type solutions can be used to validate the present higher order theory, more general boundary conditions will require solution strategies involving, e.g., boundary discontinuous double Fourier series approach.

Solution functions that completely satisfy the boundary conditions in Equations. (17) are assumed as follows:

$$u_0(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} \cos\alpha x \sin\beta y e^{-i\omega t},\ 0 \leq x \leq a;\ 0 \leq y \leq b; \quad (17a)$$

$$v_0(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn} \sin\alpha x \cos\beta y e^{-i\omega t},\ 0 \leq x \leq a;\ 0 \leq y \leq b; \quad (17b)$$

$$w_0(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin\alpha x \sin\beta y\ e^{-i\omega t},\ 0 \leq x \leq a;\ 0 \leq y \leq b; \quad (17c)$$

$$\theta_x(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} X_{mn} \cos\alpha x \sin\beta y\ e^{-i\omega t},\ 0 \leq x \leq a;\ 0 \leq y \leq b; \quad (17d)$$

$$\theta_y(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn} \sin\alpha x \cos\beta y\ e^{-i\omega t},\ 0 \leq x \leq a;\ 0 \leq y \leq b; \quad (17e)$$

$$\theta_z(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Z_{mn} \sin\alpha x \sin\beta y\ e^{-i\omega t},\ 0 \leq x \leq a;\ 0 \leq y \leq b; \quad (17f)$$

$$u_0^*(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn}^* \cos\alpha x \sin\beta y\ e^{-i\omega t},\ 0 \leq x \leq a;\ 0 \leq y \leq b; \quad (17g)$$

$$v_o^*(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn}^* \sin\alpha x \cos\beta y\ e^{-i\omega t},\ 0 \leq x \leq a;\ 0 \leq y \leq b; \quad (17h)$$

$$w_0^*(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn}^* \sin\alpha x \sin\beta y\ e^{-i\omega t},\ 0 \leq x \leq a;\ 0 \leq y \leq b; \quad (17i)$$

$$\theta_x^*(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} X_{mn}^* \cos\alpha x \sin\beta y\ e^{-i\omega t},\ 0 \leq x \leq a;\ 0 \leq y \leq b; \quad (17j)$$

$$\theta_y^*(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^* \sin\alpha x \cos\beta y\ e^{-i\omega t},\ 0 \leq x \leq a;\ 0 \leq y \leq b; \quad (17k)$$

$$\theta_z^*(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Z_{mn}^* \sin\alpha x \sin\beta y\ e^{-i\omega t},\ 0 \leq x \leq a;\ 0 \leq y \leq b \quad (17l)$$

where $\alpha = \dfrac{m\pi}{a}$ and $\beta = \dfrac{n\pi}{b}$ and m and n are modes numbers, and ω is the natural frequency of the system.

Substituting Eq. (17a)–(17l) in to Eq. (10) and collecting the coefficients one obtains

$$\left[\begin{matrix}
S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} & S_{1,5} & S_{1,6} & S_{1,7} & S_{1,8} & S_{1,9} & S_{1,10} & S_{1,11} & S_{1,12} \\
S_{2,1} & S_{2,2} & S_{2,3} & S_{2,4} & S_{2,5} & S_{2,6} & S_{2,7} & S_{2,8} & S_{2,9} & S_{2,10} & S_{2,11} & S_{2,12} \\
S_{3,1} & S_{3,2} & S_{3,3} & S_{3,4} & S_{3,5} & S_{3,6} & S_{3,7} & S_{3,8} & S_{3,9} & S_{3,10} & S_{3,11} & S_{3,12} \\
S_{4,1} & S_{4,2} & S_{4,3} & S_{4,4} & S_{4,5} & S_{4,6} & S_{4,7} & S_{4,8} & S_{4,9} & S_{4,10} & S_{4,11} & S_{4,12} \\
S_{5,1} & S_{5,2} & S_{5,3} & S_{5,4} & S_{5,5} & S_{5,6} & S_{5,7} & S_{5,8} & S_{5,9} & S_{5,10} & S_{5,11} & S_{5,12} \\
S_{6,1} & S_{6,2} & S_{6,3} & S_{6,4} & S_{6,5} & S_{6,6} & S_{6,7} & S_{6,8} & S_{6,9} & S_{6,10} & S_{6,11} & S_{6,12} \\
S_{7,1} & S_{7,2} & S_{7,3} & S_{7,4} & S_{7,5} & S_{7,6} & S_{7,7} & S_{7,8} & S_{7,9} & S_{7,10} & S_{7,11} & S_{7,12} \\
S_{8,1} & S_{8,2} & S_{8,3} & S_{8,4} & S_{8,5} & S_{8,6} & S_{8,7} & S_{8,8} & S_{8,9} & S_{8,10} & S_{8,11} & S_{8,12} \\
S_{9,1} & S_{9,2} & S_{9,3} & S_{9,4} & S_{9,5} & S_{9,6} & S_{9,7} & S_{9,8} & S_{9,9} & S_{9,10} & S_{9,11} & S_{9,12} \\
S_{10,1} & S_{102} & S_{10,3} & S_{10,4} & S_{10,5} & S_{10,6} & S_{10,7} & S_{10,8} & S_{10,9} & S_{10,10} & S_{10,11} & S_{10,12} \\
S_{11,1} & S_{11,2} & S_{11,3} & S_{11,4} & S_{11,5} & S_{11,6} & S_{11,7} & S_{11,8} & S_{11,9} & S_{11,10} & S_{11,11} & S_{11,12} \\
S_{12,1} & S_{12,2} & S_{12,3} & S_{12,4} & S_{12,5} & S_{12,6} & S_{12,7} & S_{12,8} & s_{12,9} & S_{12,10} & S_{12,11} & S_{12,12}
\end{matrix}\right] -$$

$$\omega^2 \left[\begin{matrix}
m_{1,1} & 0 & 0 & m_{1,4} & 0 & 0 & m_{1,7} & 0 & 0 & m_{1,10} & 0 & 0 \\
0 & m_{2,2} & 0 & 0 & m_{2,5} & 0 & 0 & m_{2,8} & 0 & 0 & m_{2,11} & 0 \\
0 & 0 & m_{3,3} & 0 & 0 & m_{3,6} & 0 & 0 & m_{3,9} & 0 & 0 & m_{3,12} \\
m_{4,1} & 0 & 0 & m_{4,4} & 0 & 0 & m_{4,7} & 0 & 0 & m_{4,10} & 0 & 0 \\
0 & m_{5,2} & 0 & 0 & m_{5,5} & 0 & 0 & m_{5,8} & 0 & 0 & m_{5,11} & 0 \\
0 & 0 & m_{6,3} & 0 & 0 & m_{6,6} & 0 & 0 & m_{6,9} & 0 & 0 & m_{6,12} \\
m_{7,1} & 0 & 0 & m_{7,4} & 0 & 0 & m_{7,7} & 0 & 0 & m_{7,10} & 0 & 0 \\
0 & m_{8,2} & 0 & 0 & m_{8,5} & 0 & 0 & m_{8,8} & 0 & 0 & m_{8,11} & 0 \\
0 & 0 & m_{9,3} & 0 & 0 & m_{9,6} & 0 & 0 & m_{9,9} & 0 & 0 & m_{9,12} \\
m_{10,1} & 0 & 0 & m_{10,4} & 0 & 0 & m_{10,7} & 0 & 0 & m_{10,10} & 0 & 0 \\
0 & m_{11,2} & 0 & 0 & m_{11,5} & 0 & 0 & m_{11,8} & 0 & 0 & m_{11,11} & 0 \\
0 & 0 & m_{12,3} & 0 & 0 & m_{12,6} & 0 & 0 & m_{12,9} & 0 & 0 & m_{12,12}
\end{matrix}\right] \left\{\begin{matrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ X_{mn} \\ Y_{mn} \\ Z_{mn} \\ U_{mn}^* \\ V_{mn}^* \\ W_{mn}^* \\ X_{mn}^* \\ Y_{mn}^* \\ Z_{mn}^* \end{matrix}\right\} = \left\{\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right\}$$

$$(18)$$

We may obtain the natural frequencies and vibration modes for the plate problem, by solving the eigen problem $([S] - \omega^2[M])\mathbf{X} = 0$ where $\mathbf{X}$ are the modes of vibration associated with the natural frequencies defined as ω.

4. Validation

In this section, a numerical examples are presented and discussed to validate the accuracy of the present higher-order shear deformation theory in predicting the natural frequencies of a simply supported functionally graded material plate.

Example 1: For numerical results, aluminum (Al)/zirconia(ZrO$_2$) plate is considered and graded from aluminum (as metal) at the bottom to zirconia (as ceramic) at the top surface of the plate. The material properties adopted here are

Al: Young's modulus (E_m): 70 GPa, density (ρ_m) = 2702 kg/m^3, and Poisson's ratio (υ): 0.3.

ZrO$_2$: Young's modulus (E_c): 200 GPa, density (ρ_c) = 5700 kg/m^3, and Poisson's ratio (υ): 0.3.

Presently computed results for different values of volume fraction exponent n and side-to-thickness ratos (a/h) are compared with those of Qian et al. [51] and Vel and Batra [31] (here considered to be the exact solution) and presented in Tables 1 and 2. For convenience, natural frequency ω has been nondimensionalized as $\bar{\omega} = \omega h \sqrt{\rho_m / E_m}$.

Table 1. Comparison of thickness mode nondimensional natural frequencies of a simply supported square Al/ZrO2 FG thick plate with m = 1 and n = 1 and different values of side-to-thickness ratios (a/h)

Thickness mode	a/h = 20			a/h = 10			a/h = 5		
	Present	Qian et al. [51]	Exact [31]	Present	Qian et al. [51]	Exact [31]	Present	Qin et al. [51]	Exact [31]
1	0.0158	0.0149	0.0153	0.0619	0.0584	0.0596	0.2285	0.2152	0.2192
2	0.1534	0.1457	0.1456	0.3068	0.2913	0.2912	0.6136	0.582	0.5823
3	0.2592	0.2448	0.2454	0.5179	0.4872	0.4901	1.0310	0.9687	0.9752
4	2.2140	2.0334	2.0598	2.2294	2.0788	2.075	2.2897	2.1696	2.1346

Table 2. Comparison of thickness mode nondimensional natural frequencies of a simply supported square Al/ZrO2 FG thick plate with m = 1 and n = 1, and different values of n

Thickness mode	n = 2			n = 3			n = 5		
	Present	Qin et al. [51]	Exact [31]	Present	Qin et al. [51]	Exact [31]	Present	Qin et al. [51]	Exact [31]
1	0.2264	0.2153	0.2197	0.2271	0.2172	0.2211	0.2281	0.2194	0.2225
2	0.5988	0.5709	0.5711	0.5897	0.5659	0.566	0.5790	0.5612	0.561
3	1.0056	0.9494	0.9564	0.9896	0.9414	0.9478	0.9711	0.9346	0.9398
4	2.1394	2.0154	2.015	2.0433	1.9586	1.953	1.9574	1.9204	1.9075

The presently computed nondimensionalized natural frequency of the fourth thickness mode corresponding to $m = n = 1$ and for $a/h = 20$ listed in Table 1 differs from the exact value by 7.49%; the difference is less for the lower order thickness modes. For $a/h = 5$, the difference between the computed and the analytical frequencies of the fourth thickness mode with $m = n = 1$ is 7.26%. This is due to less polynomial terms in the proposed model. This difference can be decreased and the exact value may be approached by increasing the polynomial order.

Table 2 shows the nondimensional natural frequencies of different thickness modes and volume fraction exponents. From Table 2, it is observed that the differenece is increasing with the order thickness modes. However, the side-to-thickness rato a/h of the plate has a noticeable influence on the natural frequencies.

Example 2: Table 3 shows the comparison of nondimensional natural frequencies of a square plate made of Al/ZrO$_2$ for various values of volume fraction exponent n and for a side to thickness ratio $a/h = 5$. The material properties presented in Example 1 are used for numerical results. Presently computed results for different values of volume fraction exponent n are compared with those of Neves et al. [49,50].

The results presented in Tables 1, 2 and 3 are in good agreement with those of Qian et al. [51] and Vel and Batra [31] and Neves et al. [49,50] respectively and should serve as bench mark results for future comparions.

4.1 Properties of FGPs

Aluminium (Al) Young's modulus (E_m): 70 GPa, density (ρ_m) = 2702 kg/m^3, and Poisson's ratio (υ): 0.3.

Alumina (Al$_2$O$_3$) Young's modulus (E_c): 380 GPa, density (ρ_c) = 3800 kg/m^3, and Poisson's ratio (υ): 0.3.

After establishing the accuracy of the present results for FGPs, the effect of side-to-thickness ratio (a/h), aspect ratio (a/b) and modulus ratio (E_m/E_c) on nondimen-

sional natural frequency is studied for the above material properties.

Figures 2–4 show the variation of the nondimensional natural frequency for various power law exponents "n" and with different side-to-thickness ratios (a/h), aspect ratios (a/b) and modulus ratios (E_m/E_c) respectively according to present higher-order shear deformation theory. From Figure 2 it is clear that the bending-stretching coupling and transverse shear deformation effect is decreasing frequencies is felt for $a/h \leq 10$ for simply supported boundary conditions. The effect of shear deformation decreases with the increasing values of a/h and decreasing values of volume fraction exponent. In Figure 3, it is observed that, the effect of coupling is to decrease the natural frequencies for lower values of aspect ratio. The coupling is maximum for metals and minimum for ceramics. Finally, Figure 4 depicts the variation of fundamental frequency for different modulus ratios and volume fraction exponents. It can be seen that, the effect of coupling is significant for all
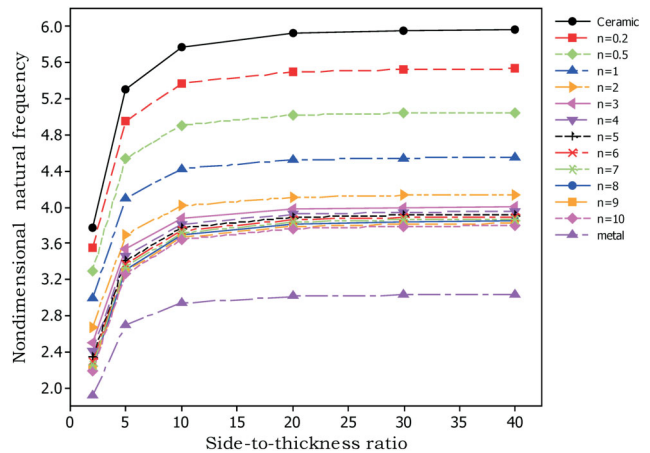

Figure 2. Effect of side-to-thickness ratios on the nondimensional natural frequency ($\overline{\omega} = \omega \dfrac{a^2}{h} \sqrt{\rho_c / E_c}$) of an FGM plate for different values of volume fraction exponents (n).

Table 3. Fundamental frequency of square FG plate (Al/ZrO$_2$) with $a/h = 5$ ($\overline{\omega} = \omega h \sqrt{\rho_m / E_m}$)

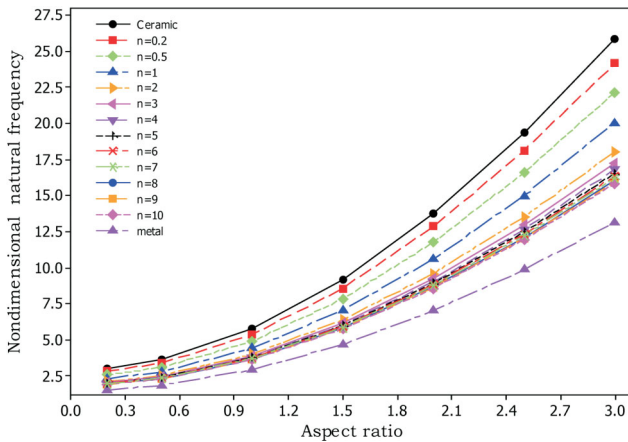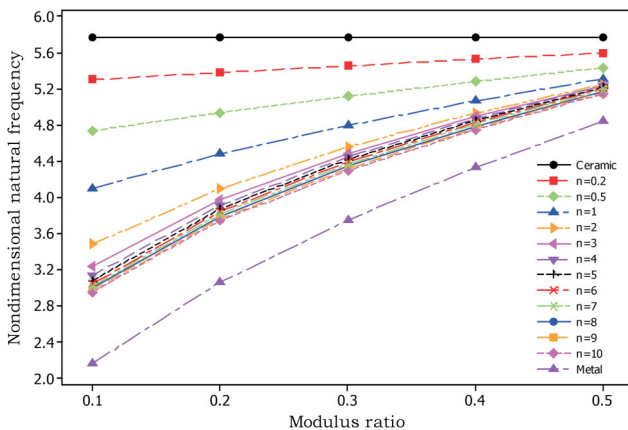| $a/h = 5$	source	Power law index (n)						
		0	0.5	1	2	3	5	10
	Ref. [49]			0.2193	0.2198	0.2212	0.2225	
	Ref. [48]	0.247	0.2228	0.2193	0.2200	0.2215	0.2230	0.2229
	Present	0.247	0.234	0.2285	0.2264	0.2271	0.2281	0.2272

Figure 3. Effect of aspect ratios on the nondimensional natural frequency ($\overline{\omega} = \omega \dfrac{a^2}{h} \sqrt{\rho_c / E_c}$) of an FGM plate for different values of volume fraction exponents (*n*).

Figure 4. Effect of anisotropy on the nondimensional natural frequency ($\overline{\omega} = \omega \dfrac{a^2}{h} \sqrt{\rho_c / E_c}$) of an FGP for different values of volume fraction exponents (*n*).

modulus ratios and volume fraction exponents.

5. Conclusions

Analytical formulations and solutions for vibration analysis of functionally graded material plates is developed using a higher-order shear deformation theory considering the ε_z which account for transverse extensibility and without enforcing zero shear on the top and bottom of the FGPs. Equations of motion are derived from the Hamilton's principle. Closed form solutions are obtained for simply supported plates using Naviers me-

thod and solving the eigen value problem. The accuracy and efficiency of the present theory have been demonstrated in the vibration behavior of FGPs. The results are compared with the other higher order shear deformation theory. The present results are in good agreement with those of Qian et al. [51] and Vel and Batra [31] and Neves et al. [49,50] and should serve as bench mark results for future comparisons. In conclusion, it can be said that the proposed theory is accurate and simple in analyzing the vibration behavior of FGPs.

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