A Feedback Retrial Queuing System with Starting Failures and Single Vacation

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Abstract

The M/G/1 retrial queue with Bernoulli feedback and single vacation is studied in this paper, where the server is subjected to starting failure. The retrial time is assumed to follow an arbitrary distribution and the customers in the orbit access the server under FCFS discipline. The server leaves for a vacation as soon as the system becomes empty. When the server returns from the vacation and finds no customers, he waits free for the first customer to arrive from outside the system. The system size distribution at random points and various performance measures are derived. The general decomposition law is shown to hold good for this model also. Some of the existing results in [7] are deduced as special cases from our results.

***Key Words*:** Feedback, Vacation, Starting Failures, Retrial Queues, Steady State

1. Introduction

Retrial queues have been widely used to model many problems in telephone switching systems, telecommunication networks and computers competing to gain service from a central processor unit. Most of the papers on retrial queues have considered the system without feedback. A more practical retrial queue with feedback occurs in many practical situations: for example, multiple access telecommunication systems, where messages turned out as errors are sent again, can be modeled as retrial queue with feedback. Choi and Kulkarni [1] have studied M/G/1 retrial queue with feedback. A remarkable and unavoidable phenomenon in the service facility of a queuing system is its breakdown. Kulkarni and Choi [2] have analysed the M/G/1 retrial queue with server subjected to repairs and breakdowns. Further, Choi et al. [3] have investigated the M/G/1 retrial queue with customer collisions. Recently, Yang and Li [4] have studied the M/G/1 retrial queue with the server subject to starting failures. They obtained analytical results for the queue length distribution and the stochastic decomposition law, where the retrial time is assumed to be exponentially distributed. Fayolle [5] has investigated an M/M/1 retrial queue where the customers in the retrial group form a queue and only the customer at the head of the queue can request a service to the server after exponentially distributed retrial time with constant rate. A retrial queuing system with FCFS discipline and general retrial times has been extensively discussed by Gomez-Cortal [6]. Krishna, Pavai and Vijayakumar [7] discussed a retrial queue with feedback and starting failures.

In this paper, we consider an M/G/1 retrial queue with Bernoulli feedback and single vacation where the server is subjected to starting failures. We also assume that the retrial time is governed by an arbitrary distribution and that the customer at the head of the orbit queue is allowed for access to the server.

2. Model Description

We consider a single-server retrial queue with the

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server being subjected to starting failures and single vacation. New customers arrive from outside the system according to a Poisson process with rate λ. We assume that there is no waiting space and therefore if an arriving customer finds the server busy or down, the customer is obliged to leave the service area and repeat his request for service after some random time. Between trials, the blocked customer joins a pool of unsatisfied customers called "orbit" in accordance with an FCFS discipline. That is, only the customer at the head of the orbit queue is allowed for access to the server. Successive inter-retrial times of any customer are governed by an arbitrary probability distribution function A(x), with corresponding density function a(x) and Laplace-Stieltjes transform $\gamma^*(\theta)$. If the server is free, an arriving (primary or retrial) customer must start (turn on) the server, which takes negligible time. If the server is started successfully (with a certain probability), the customer gets service immediately. Otherwise, the "repair" for the server commences immediately and the customer must leave for the orbit and make a retrial at a later time. Successive service times are independent with common probability distribution function B(x), density function b(x), Laplace-Stieltjes transform $\beta^*(\theta)$ and first three moments β_1, β_2 and β_3. Similarly, the successive repair times are independent and identically distributed with probability distribution function D(x), density function d(x), Laplace-Stieltjes transform $\varphi^*(\theta)$ and first three moments φ_1, φ_2 and φ_3. As soon as the system becomes empty, the server leaves the system for a vacation period of random length. On returning from the vacation, the server either finds no customers and waits free until the arrival of a primary customer, or finds at least one customer and begins service. Vacation times are independent with common probability distribution function G(x), density function g(x), Laplace-Stieltjes transform $\psi^*(\theta)$ and first three moments ψ_1, ψ_2 and ψ_3.

After the customer is served completely, he will decide either to join the retrial group again for another service with probability p or to leave the system forever with probability q (= 1 – p). It is assumed that the probability of successful commencement of service is δ for a new customer who finds the server free and sees no other customer in the orbit (the counterpart of a customer who starts a busy period in the standard M/G/1 system) and is α for all other new and returning customers. Interarrival times, retrial times, service times, vacation times and breakdown times are assumed to be mutually independent. From this description, it is clear that at any service completion, the server becomes free and in such a case, a possible new (primary) arrival and the one (if any) at the head of the orbit queue, compete for service.

The state of the system at time t can be described by the Markov process $\{N(t);\ t \geq 0\} = \{(C(t), X(t), \xi_0(t), \xi_1(t), \xi_2(t), \xi_3(t)),\ t \geq 0\}$, where C(t) denotes the server state (0, 1, 2 or 3 according as the server being free, busy, repair or on vacation out of the service facility, respectively) and X(t) corresponds to the number of customers in orbit at time t. If C(t) = 0 and X(t) > 0, then $\xi_0(t)$ respresents the elapsed retrial time, if C(t) = 1, then $\xi_1(t)$ corresponds to the elapsed time of the customer being served, if C(t) = 2 and X(t) > 0, then $\xi_2(t)$ represents the elapsed repair time at time t, if C(t) = 3 and X(t) $\geq$ 0, then $\xi_3(t)$ represents the elapsed vacation time at time t. The functions r(x), $\mu(x)$, $\eta(x)$ and $\nu(x)$ are the conditional completion rates (at time x) for repeated attempts, for service, for repair and for vacation, respectively; i.e., $r(x) = a(x)\,(1 - A(x))^{-1}$, $\mu(x) = b(x)\,(1 - B(x))^{-1}$, $\eta(x) = d(x)\,(1 - D(x))^{-1}$ and $\nu(x) = g(x)\,(1 - G(x))^{-1}$.

3. Steady State Distribution

For the process $\{X(t);\ t \geq 0\}$ we define the probability

$P_0(t) = p\ \{C(t) = 0, X(t) = 0\}$,

$P_0 = \lim_{t\to\infty} P_0(t)$;

and the probability densities

$P_n(x, t)\,dx = p\ \{C(t) = 0, X(t) = n, x \leq \xi_0(t) < x + dx\}$, for $t \geq 0$, $x \geq 0$ and $n \geq 1$,

$P_n(x) = \lim_{t\to\infty} P_n(x, t)$, for $x \geq 0$ and $n \geq 1$;

$Q_n(x, t)\,dx = p\ \{C(t) = 1, X(t) = n, x \leq \xi_1(t) < x + dx\}$, for $t \geq 0$, $x \geq 0$ and $n \geq 0$,

$Q_n(x) = \lim_{t\to\infty} Q_n(x, t)$, for $x \geq 0$ and $n \geq 0$;

$R_n(x, t)\,dx = p\ \{C(t) = 2, X(t) = n, x \leq \xi_2(t) < x + dx\}$, for $t \geq 0$, $x \geq 0$ and $n \geq 1$,

$R_n(x) = \lim_{t\to\infty} R_n(x, t)$, for $x \geq 0$ and $n \geq 1$;

and

$W_n(x, t)\,dx = p\{C(t) = 3, X(t) = n, x \le \xi_3(t) < x + dx\}$, for $t \ge 0$, $x \ge 0$ and $n \ge 0$,

$W_n(x) = \lim_{t\to\infty} W_n(x, t)$, for $x \ge 0$ and $n \ge 0$.

By the method of supplementary variable technique, we obtain the following system of equations that govern the dynamics of the system behavior:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \int_0^\infty W_0(x,t)\nu(x)dx \tag{3.1}$$

$$\frac{\partial P_n(x,t)}{\partial t} + \frac{\partial P_n(x,t)}{\partial x} = -[\lambda + r(x)]P_n(x,t),\ n \ge 1 \tag{3.2}$$

$$\frac{\partial Q_0(x,t)}{\partial t} + \frac{\partial Q_0(x,t)}{\partial x} = -[\lambda + \mu(x)]Q_0(x,t) \tag{3.3}$$

$$\frac{\partial Q_n(x,t)}{\partial t} + \frac{\partial Q_n(x,t)}{\partial x} = -[\lambda + \mu(x)]Q_n(x,t) + \lambda Q_{n-1}(x,t),\ n \ge 1 \tag{3.4}$$

$$\frac{\partial R_1(x,t)}{\partial t} + \frac{\partial R_1(x,t)}{\partial x} = -[\lambda + \eta(x)]R_1(x,t) \tag{3.5}$$

$$\frac{\partial R_n(x,t)}{\partial t} + \frac{\partial R_n(x,t)}{\partial x} = -[\lambda + \eta(x)]R_n(x,t) + \lambda R_{n-1}(x,t),\ n \ge 2 \tag{3.6}$$

$$\frac{\partial W_0(x,t)}{\partial t} + \frac{\partial W_0(x,t)}{\partial x} = -[\lambda + \nu(x)]W_0(x,t) \tag{3.7}$$

$$\frac{\partial W_n(x,t)}{\partial t} + \frac{\partial W_n(x,t)}{\partial x} = -[\lambda + \nu(x)]W_n(x,t) + \lambda W_{n-1}(x,t),\ n \ge 1 \tag{3.8}$$

with the boundary conditions

$$P_n(0,t) = q\int_0^\infty Q_n(x,t)\mu(x)dx + p\int_0^\infty Q_{n-1}(x,t)\mu(x)dx + \int_0^\infty R_n(x,t)\eta(x)dx + \int_0^\infty W_n(x,t)\nu(x)dx,\ n \ge 1 \tag{3.9}$$

$$Q_0(0,t) = \delta\lambda P_0(t) + \alpha\int_0^\infty P_1(x,t)r(x)dx \tag{3.10}$$

$$Q_n(0,t) = \alpha\lambda\int_0^\infty P_n(x,t)dx + \alpha\int_0^\infty P_{n+1}(x,t)r(x)dx,\ n \ge 1 \tag{3.11}$$

$$R_1(0,t) = \bar{\delta}\lambda P_0(t) + \bar{\alpha}\int_0^\infty P_1(x,t)r(x)dx \tag{3.12}$$

$$R_n(0,t) = \bar{\alpha}\lambda\int_0^\infty P_{n-1}(x,t)dx + \bar{\alpha}\int_0^\infty P_n(x,t)r(x)dx,\ n \ge 2 \tag{3.13}$$

$$W_0(0,t) = q\int_0^\infty Q_0(x,t)\mu(x)dx \tag{3.14}$$

$$W_n(0, t) = 0,\ n \ge 1 \tag{3.15}$$

and the normalizing condition

$$P_0(t) + \sum_{n=1}^\infty\int_0^\infty P_n(x,t)dx + \sum_{n=0}^\infty\int_0^\infty Q_n(x,t)dx + \sum_{n=1}^\infty\int_0^\infty R_n(x,t)dx + \sum_{n=0}^\infty\int_0^\infty W_n(x,t)dx = 1 \tag{3.16}$$

Letting $t \to \infty$ in equations (3.1)–(3.16), one has

$$\lambda P_0 = \int_0^\infty W_0(x)\nu(x)dx \tag{3.17}$$

$$\frac{dP_n(x)}{dx} = -[\lambda + r(x)]P_n(x),\ n \ge 1 \tag{3.18}$$

$$\frac{dQ_0(x)}{dx} = -[\lambda + \mu(x)]Q_0(x) \tag{3.19}$$

$$\frac{dQ_n(x)}{dx} = -[\lambda + \mu(x)]Q_n(x) + \lambda Q_{n-1}(x),\ n \ge 1 \tag{3.20}$$

$$\frac{dR_1(x)}{dx} = -[\lambda + \eta(x)]R_1(x) \tag{3.21}$$

$$\frac{dR_n(x)}{dx} = -[\lambda + \eta(x)]R_n(x) + \lambda R_{n-1}(x),\ n \ge 2 \tag{3.22}$$

$$\frac{dW_0(x)}{dx} = -[\lambda + \nu(x)]W_0(x) \tag{3.23}$$

$$\frac{dW_n(x)}{dx} = -[\lambda + \nu(x)]W_n(x) + \lambda W_{n-1}(x),\ n \geq 1 \tag{3.24}$$

The steady-state boundary conditions are

$$P_n(0) = q\int_0^\infty Q_n(x)\mu(x)dx + p\int_0^\infty Q_{n-1}(x)\mu(x)dx + \int_0^\infty R_n(x)\eta(x)dx + \int_0^\infty W_n(x)\nu(x)dx,\ n \geq 1 \tag{3.25}$$

$$Q_0(0) = \delta\lambda P_0 + \alpha\int_0^\infty P_1(x)r(x)dx \tag{3.26}$$

$$Q_n(0) = \alpha\lambda\int_0^\infty P_n(x)dx + \alpha\int_0^\infty P_{n+1}(x)r(x)dx,\ n \geq 1 \tag{3.27}$$

$$R_1(0) = \overline{\delta}\lambda P_0 + \overline{\alpha}\int_0^\infty P_1(x)r(x)dx \tag{3.28}$$

$$R_n(0) = \overline{\alpha}\lambda\int_0^\infty P_{n-1}(x)dx + \overline{\alpha}\int_0^\infty P_n(x)r(x)dx,\ n \geq 2 \tag{3.29}$$

$$W_0(0) = q\int_0^\infty Q_0(x)\mu(x)dx \tag{3.30}$$

$$W_n(0) = 0,\ n \geq 1 \tag{3.31}$$

and the normalizing condition is

$$P_0 + \sum_{n=1}^\infty\int_0^\infty P_n(x)dx + \sum_{n=0}^\infty\int_0^\infty Q_n(x)dx + \sum_{n=1}^\infty\int_0^\infty R_n(x)dx + \sum_{n=0}^\infty\int_0^\infty W_n(x)dx = 1 \tag{3.32}$$

To solve equations (3.17)–(3.32), we define the generating functions

$$P(x,z) = \sum_{n=1}^\infty z^n P_n(x),\quad Q(x,z) = \sum_{n=0}^\infty z^n Q_n(x),$$

$$R(x,z) = \sum_{n=1}^\infty z^n R_n(x) \text{ and } W(x,z) = \sum_{n=0}^\infty z^n W_n(x)$$

The following theorem discusses the steady state distribution of the system.

Theorem 3.1:

The joint steady state distribution of $\{X(t); t \geq 0\}$ is obtained as

$$P(x,z) = \frac{P_0\lambda z\{(pz+q)\delta\beta^*(\lambda(1-z)) + z\overline{\delta}\varphi^*(\lambda(1-z)) - 1\} - [1-\psi^*(\lambda(1-z))]zW(0,z)}{z - \{(pz+q)\alpha\beta^*(\lambda(1-z)) + z\overline{\alpha}\varphi^*(\lambda(1-z))\}} \{z + (1-z)\gamma^*(\lambda)\} \times e^{-\lambda x - \int_0^x r(u)du} \tag{3.33}$$

$$Q(x,z) = \Big[P_0\delta\lambda + \Big\{P_0\lambda\alpha[z + (1-z)\gamma^*(\lambda)][(pz+q)\delta\beta^*(\lambda(1-z)) + z\overline{\delta}\varphi^*(\lambda(1-z)) - 1] - \alpha[1-\psi^*(\lambda(1-z))][z + (1-z)\gamma^*(\lambda)]W(0,z)\Big\} \Big/ \Big\{z - [(pz+q)\alpha\beta^*(\lambda(1-z)) + z\overline{\alpha}\varphi^*(\lambda(1-z))][z + (1-z)\gamma^*(\lambda)]\Big\}\Big] \times e^{-\lambda(1-z)x - \int_0^x \mu(u)du} \tag{3.34}$$

$$R(x,z) = \Big[P_0\overline{\delta}\lambda z + \Big\{P_0\overline{\alpha}\lambda z[z + (1-z)\gamma^*(\lambda)][(pz+q)\delta\beta^*(\lambda(1-z)) + z\overline{\delta}\varphi^*(\lambda(1-z)) - 1] - \overline{\alpha}z[1-\psi^*(\lambda(1-z))][z + (1-z)\gamma^*(\lambda)]W(0,z)\Big\} \Big/ \Big\{z - [(pz+q)\alpha\beta^*(\lambda(1-z)) + z\overline{\alpha}\varphi^*(\lambda(1-z))][z + (1-z)\gamma^*(\lambda)]\Big\}\Big] \times e^{-\lambda(1-z)\,x - \int_0^x \eta(u)du} \tag{3.35}$$

$$W(x,z) = W(0,z)e^{-\lambda\,(1-z)x - \int_0^x \nu(u)du} \tag{3.36}$$

where

$$W(0,z) = q\int_0^\infty Q_0(x)\mu(x)dx = \lambda P_0/\psi^*(\lambda) \quad \text{(see [8])}$$

$$\beta^*(\lambda(1-z)) = \int_0^\infty \mu(x)e^{-\lambda(1-z)x}e^{-\int_0^x \mu(u)du}dx$$

$$\varphi^*(\lambda(1-z)) = \int_0^\infty \eta(x)e^{-\lambda(1-z)x}e^{-\int_0^x \eta(u)du}dx$$

$$\psi^*(\lambda(1-z)) = \int_0^\infty \nu(x)e^{-\lambda(1-z)x}e^{-\int_0^x \nu(u)du}dx$$

$$\gamma^*(\lambda) = \int_0^\infty r(x)e^{-\lambda x}e^{-\int_0^x r(u)du}dx$$

and the probability P_0 can be determined from the normalization condition.

Proof:

Multiplying equations (3.17)–(3.31) by z^n and summing over n, n = 0, 1, 2, ... one can obtain the following equations:

$$\frac{\partial P(x,z)}{\partial x} + [\lambda + r(x)]P(x,z) = 0 \tag{3.37}$$

$$\frac{\partial Q(x,z)}{\partial x} + [\lambda(1-z) + \mu(x)]Q(x,z) = 0 \tag{3.38}$$

$$\frac{\partial R(x,z)}{\partial x} + [\lambda(1-z) + \eta(x)]R(x,z) = 0 \tag{3.39}$$

$$\frac{\partial W(x,z)}{\partial x} + [\lambda(1-z) + \nu(x)]W(x,z) = 0 \tag{3.40}$$

$$P(0,z) = (pz+q)\int_0^\infty Q(x,z)\mu(x)dx + \int_0^\infty R(x,z)\eta(x)dx + \int_0^\infty W(x,z)\nu(x)dx - q\int_0^\infty Q_0(x)\mu(x)dx - \lambda P_0 \tag{3.41}$$

$$Q(0,z) = \delta\lambda P_0 + \alpha\lambda\int_0^\infty P(x,z)dx + \frac{\alpha}{z}\int_0^\infty P(x,z)r(x)dx \tag{3.42}$$

$$R(0,z) = z\lambda\bar{\delta}P_0 + \bar{\alpha}\lambda z\int_0^\infty P(x,z)dx + \bar{\alpha}\int_0^\infty P(x,z)r(x)dx \tag{3.43}$$

$$W(0,z) = q\int_0^\infty Q_0(x)\mu(x)dx \tag{3.44}$$

Solving the partial differential equations (3.37)–(3.40), to obtain

$$P(x,z) = P(0,z)e^{-\lambda x - \int_0^x r(u)du} \tag{3.45}$$

$$Q(x,z) = Q(0,z)e^{-\lambda(1-z)x - \int_0^x \mu(u)du} \tag{3.46}$$

$$R(x,z) = R(0,z)e^{-\lambda(1-z)x - \int_0^x \eta(u)du} \tag{3.47}$$

$$W(x,z) = W(0,z)e^{-\lambda(1-z)x - \int_0^x \nu(u)du} \tag{3.48}$$

Using (3.46)–(3.48) in (3.41) and solving for P(0,z), after some manipulations, one can get

$$P(0,z) = \frac{P_0\lambda z\{(pz+q)\delta\beta^*(\lambda(1-z)) + z\bar{\delta}\varphi^*(\lambda(1-z)) - 1\} - [1-\psi^*(\lambda(1-z))]zW(0,z)}{z - [(pz+q)\alpha\beta^*(\lambda(1-z)) + z\bar{\alpha}\varphi^*(\lambda(1-z))][z+(1-z)\gamma^*(\lambda)]} \tag{3.49}$$

Combining (3.45), (3.49) and (3.42) and on simplification we have

$$Q(0,z) = P_0\delta\lambda + \Big\{P_0\lambda\alpha[z+(1-z)\gamma^*(\lambda)][(pz+q)\delta\beta^*(\lambda(1-z)) + z\bar{\delta}\varphi^*(\lambda(1-z)) - 1] - \alpha[1-\psi^*(\lambda(1-z))][z+(1-z)\gamma^*(\lambda)]W(0,z)\Big\} \Big/ \Big\{z - [(pz+q)\alpha\beta^*(\lambda(1-z)) + z\bar{\alpha}\varphi^*(\lambda(1-z))][z+(1-z)\gamma^*(\lambda)]\Big\} \tag{3.50}$$

Similarly, substituting (3.45) and (3.49) into (3.43) we get

$$R(0,z) = P_0\bar{\delta}\lambda z + \Big\{P_0\bar{\alpha}\lambda z[z+(1-z)\gamma^*(\lambda)] [(pz+q)\delta\beta^*(\lambda(1-z)) + z\bar{\delta}\varphi^*(\lambda(1-z)) - 1] - \bar{\alpha}z[1-\psi^*(\lambda(1-z))][z+(1-z)\gamma^*(\lambda)]W(0,z)\Big\} \Big/ \Big\{z - [(pz+q)\alpha\beta^*(\lambda(1-z)) + z\bar{\alpha}\varphi^*(\lambda(1-z))] [z+(1-z)\gamma^*(\lambda)]\Big\} \quad (3.51)$$

Combining (3.45)–(3.51), we obtain the required results (3.33)–(3.35).

For the limiting probability generating functions P(x,z), Q(x,z), R(x,z) and W(x,z) define $P(z) = \int_0^\infty P(x,z)dx$, $Q(z) = \int_0^\infty Q(x,z)dx$, $R(z) = \int_0^\infty R(x,z)dx$, and $W(z) = \int_0^\infty W(x,z)dx$ and the probability generating function of the number of customers in the system under steady state is K(z) = P_0 + P(z) + zQ(z) + R(z) + W(z). Then the main result is given by

Theorem 3.2:

In steady state,

$$P(z) = \Big[z[1-\gamma^*(\lambda)]\Big\{\lambda P_0[(pz+q)\delta\beta^*(\lambda(1-z)) + z\bar{\delta}\varphi^*(\lambda(1-z)) - 1] - [1-\psi^*(\lambda(1-z))]W(0,z)\Big\}\Big] \Big/ \Big[\lambda\Big\{z - [(pz+q)\alpha\beta^*(\lambda(1-z)) + z\bar{\alpha}\varphi^*(\lambda(1-z))][z+(1-z)\gamma^*(\lambda)]\Big\}\Big] \quad (3.52)$$

$$Q(z) = \Big[[1-\beta^*(\lambda(1-z))]\Big\{\lambda P_0[\delta z - \{z+(1-z)\gamma^*(\lambda)\} \{(\alpha-\delta)z\varphi^*(\lambda(1-z)) + \alpha\}] - \alpha[1-\psi^*(\lambda(1-z))] [z+(1-z)\gamma^*(\lambda)]W(0,z)\Big\}\Big] \Big/ \Big[\lambda(1-z)\Big\{z - [(pz+q)\alpha\beta^*(\lambda(1-z)) + z\bar{\alpha}\varphi^*(\lambda(1-z))][z+(1-z)\gamma^*(\lambda)\Big]\Big\}\Big] \quad (3.53)$$

$$R(z) = \Big[z[1-\varphi^*(\lambda(1-z))]\Big\{\lambda P_0\Big[\bar{\delta}z - \{z+(1-z)\gamma^*(\lambda)\} \{(\alpha-\delta)(pz+q)\beta^*(\lambda(1-z)) + \bar{\alpha}\}\Big] - \bar{\alpha}z[1-\psi^*(\lambda(1-z))][z+(1-z)\gamma^*(\lambda)]W(0,z)\Big\}\Big] \Big/ \Big[z - [(pz+q)\alpha\beta^*(\lambda(1-z)) + z\bar{\alpha}\varphi^*(\lambda(1-z))][z+(1-z)\gamma^*(\lambda)]\Big\}\Big] \quad (3.54)$$

$$W(z) = \frac{[1-\psi^*(\lambda(1-z))]}{\lambda(1-z)} \cdot W(0,z) \quad (3.55)$$

and

$$K(z) = \Big[\lambda P_0\Big\{[z+(1-z)\gamma^*(\lambda)][q\beta^*(\lambda(1-z)) \{z\varphi^*(\lambda(1-z))(\alpha-\delta)-\alpha\}]\Big\} + \delta zq\beta^*(\lambda(1-z))\Big\} - q\alpha\beta^*(\lambda(1-z))[1-\psi^*(\lambda(1-z))] [z+(1-z)\gamma^*(\lambda)]W(0,z)\Big] \Big/ \Big[\lambda\Big\{z - [(pz+q)\alpha\beta^*(\lambda(1-z)) + z\bar{\alpha}\varphi^*(\lambda(1-z))][z+(1-z)\gamma^*(\lambda)]\Big\}\Big] \quad (3.56)$$

where

$$P_0 = \frac{\gamma^*(\lambda) - \alpha(\lambda\beta_1 + p) - \bar{\alpha}(1+\lambda\varphi_1) - \alpha\psi_1 qW(0,z)}{q\delta\gamma^*(\lambda) + q(\alpha-\delta)(1+\lambda\varphi_1)} \quad (3.57)$$

Proof:

Integrating the equations (3.33)–(3.36) from 0 to ∞ with respect to x, we obtain respectively (3.52)–(3.55). At this point, the only unknown is P_0 which can be determined using the normalizing condition, P_0 + P(1) + Q(1) + R(1) + W(1) = 1. Thus, setting z = 1 in (3.52)–(3.55) and applying L'Hopital's rule whenever necessary, we get after using the normalizing condition and rearrangement.

$$\frac{P_0[q\delta\gamma^*(\lambda) + q(\alpha-\delta)(1+\lambda\varphi_1)] + \alpha q\psi_1 W(0,z)}{\gamma^*(\lambda) - \alpha(\lambda\beta_1 + p) - \bar{\alpha}(1+\lambda\varphi_1)} = 1$$

which yields (3.57).

Finally, the probability generating function K(z) = P_0 +

P(z) + zQ(z) + R(z) + W(z) of the number of customers in the system is obtained by using (3.52)–(3.55) and some mathematical manipulations yield (3.56).

Remark 3.1.

If $\alpha = \delta = q = 1$, then (3.56) and (3.57) become

$$K(z) = \Big[\beta^*(\lambda(1-z))\Big\{\lambda(1-z)[\gamma^*(\lambda) - \lambda\beta_1 + \psi_1 W(0,z)] + [1-\psi^*(\lambda(1-z))][z + (1-z)\gamma^*(\lambda)]W(0,z)\Big\}\Big] \Big/ \Big[\lambda\Big\{\gamma^*(\lambda)(1-z)\beta^*(\lambda(1-z)) - z[1-\beta^*(\lambda(1-z))]\Big\}\Big] \tag{3.58}$$

and

$$P_0 = 1 - \frac{\lambda\beta_1 + \psi_1 W(0,z)}{\gamma^*(\lambda)} \tag{3.59}$$

If the vacation do not exist, the results (3.58), (3.59) will reduce to the same results in [7] which agree with Gomez-Corral [6] without feedback and starting failures.

Remark 3.2.

Further, if the retrial time distribution is exponential with parameter ν, then $\gamma^*(\lambda) = \nu/(\lambda + \nu)$. In this case (3.58) yields

$$K(z) = \Big[\beta^*(\lambda(1-z))\{\lambda(1-z)[\nu - \lambda(\lambda+\nu)\beta_1 + \psi_1(\lambda+\nu)W(0,z)] + [1-\psi^*(\lambda(1-z))](\nu+\lambda z)W(0,z)\Big\}\Big] \Big/ \Big[\lambda\Big\{\nu[\beta^*(\lambda(1-z)) - z] + \lambda z[\beta^*(\lambda(1-z)) - 1]\Big\}\Big] \tag{3.60}$$

If the vacation do not exist equation (3.60) coincides with the equation (2.13) in [9].

We now obtain some performance measures for the system in steady state. Let U be the server utilization, (or the steady state probability that the server is attending a customer) that is the server is busy, I, the steady state probability that the server is idle during the retrial time, R, the steady state probability that the server is under repair, W, the steady state probability that the server is on vacation and D, the steady probability that the server is down or free or on vacation. From Theorem 3.2, we obtain:

$$U = Q(1) = \lambda\,\beta_1/q,$$

$$I = P(1) = \frac{[1-\gamma^*(\lambda)][\lambda\delta\beta_1 - q\delta + 1 + \lambda\bar{\delta}\varphi_1] + q\delta\psi_1[1-\gamma^*(\lambda)]W(0,z)}{q\delta\gamma^*(\lambda) + q(\alpha-\delta)(1+\lambda\varphi_1)}$$

$$R = R(1) = \frac{\lambda\varphi_1\{\bar{\delta}\gamma^*(\lambda) - (\alpha-\delta)(p+\lambda\beta_1)\} - \lambda q\varphi_1\psi_1(\alpha-\delta)W(0,z)}{q\delta\gamma^*(\lambda) + (\alpha-\delta)q(1+\lambda\varphi_1)}$$

$$W = W(1) = \psi_1\, W(0, z)$$

and

$$D = P_0 + P(1) + R(1) + W(1) = 1 - \frac{\lambda\beta_1}{q}$$

The mean number of customers in the system L_S under steady state conditions is obtained by differentiating (3.56) with respect to z and evaluation at z = 1.

$$L_S = \frac{(\alpha-\delta)[\lambda^2\varphi_2 + 2\lambda\varphi_1 + 2(1+\lambda\varphi_1)\{\lambda\beta_1 + 1 - \gamma^*(\lambda)\}] + 2\lambda\beta_1\delta\gamma^*(\lambda)}{2[\delta\gamma^*(\lambda) + (\alpha-\delta)(1+\lambda\varphi_1)]} + \frac{2(1-\gamma^*(\lambda))[\alpha(\lambda\beta_1 + p) + \bar{\alpha}(1+\lambda\varphi_1)]}{2[\gamma^*(\lambda) - \alpha(\lambda\beta_1 + p) - \bar{\alpha}(1+\lambda\varphi_1)]} + \frac{+2\lambda(\alpha p\beta_1 + \bar{\alpha}\varphi_1) + \lambda^2(\alpha\beta_2 + \bar{\alpha}\varphi_2)}{2[\gamma^*(\lambda) - \alpha(\lambda\beta_1 + p) - \bar{\alpha}(1+\lambda\varphi_1)]} - \frac{\alpha q\psi_1 W(0,z)\{(\alpha-\delta)[2\lambda\varphi_1 + \lambda^2\varphi_2 + 2(1+\lambda\varphi_1) + [\lambda\beta_1 + 1 - \gamma^*(\lambda)]] + 2\lambda\beta_1\delta\gamma^*(\lambda)\}}{2[\gamma^*(\lambda) - \alpha(\lambda\beta_1 + p) - \bar{\alpha}(1+\lambda\varphi_1)][\delta\gamma^*(\lambda) + (\alpha-\delta)(1+\lambda\varphi_1)]} + \frac{\lambda\alpha q\psi_2 W(0,z) + 2q\alpha\psi_1[\lambda\beta_1 + 1 - \gamma^*(\lambda)]W(0,z)}{2\{\gamma^*(\lambda) - \alpha(\lambda\beta_1 + p) - \bar{\alpha}(1+\lambda\varphi_1)\}}$$

The orbit characteristics are of considerable interest in retrial queues. For the model under consideration, we obtain the probability of orbit being empty as

$$V = P_0 + Q_0 + W_0$$

where Q_0 is the probability that the orbit is empty while the server is busy, P_0 is the probability of an empty system and W_0 is the probability that the orbit is empty while the server is on vacation out of the system, we observe that

$$Q_0 = \frac{P_0[1-\beta^*(\lambda(1-z))]}{q\beta^*(\lambda)} + \frac{[1-\beta^*(\lambda(1-z))][1-\psi^*(\lambda(1-z))]W(0,z)}{\lambda q\beta^*(\lambda)}$$

and

$$W_0 = \frac{[1-\psi^*(\lambda)]W(0,z)}{\lambda}$$

where P_0 is given in (3.57) and

$$V = \frac{[1-p\beta^*(\lambda)][\gamma^*(\lambda)-\alpha(\lambda\beta_1+p)-\bar{\alpha}(1+\lambda\varphi_1)]}{q\beta^*(\lambda)[q\delta\gamma^*(\lambda)+(\alpha-\delta)q(1+\lambda\varphi_1)]} - \frac{\alpha\psi_1[1-p\beta^*(\lambda)]W(0,z)}{\beta^*(\lambda)[q\delta\gamma^*(\lambda)+q(\alpha-\delta)(1+\lambda\varphi_1)]} + \frac{[1-p\beta^*(\lambda)][1-\psi^*(\lambda)]W(0,z)}{\lambda q\beta^*(\lambda)}$$

Define $H(z) = P_0 + P(z) + Q(z) + R(z) + W(z)$. Then $H(z)$ represents the probability generating function for the number of customers in the orbit. Using (3.52)–(3.55) and simplifying, we get

$$H(z) = \Big[\lambda P_0\Big\{[z+(1-z)\gamma^*(\lambda)]z\varphi^*(\lambda(1-z))[1-p\beta^*(\lambda(1-z))]+pz\beta^*(\lambda(1-z))-z\Big\} (\alpha-\delta)+\lambda P_0(1-z)\gamma^*(\lambda)[\alpha p\beta^*(\lambda(1-z))-\alpha] -[1-\psi^*(\lambda(1-z))][\alpha-\alpha p\beta^*(\lambda(1-z))] [z+(1-z)\gamma^*(\lambda)]W(0,z)\Big]\Big/\Big[\lambda\Big\{z-[(pz+q)\alpha\beta^*(\lambda(1-z)) +z\bar{\alpha}\varphi^*(\lambda(1-z))][z+(1-z)\gamma^*(\lambda)]\Big\}\Big]$$

Hence, the mean number of customers in the orbit is given by

$$Lq = H`(1) = \frac{(\alpha-\delta)[2(1+\lambda\varphi_1)\{q(1-\gamma^*(\lambda))-p\lambda\beta_1\} +\lambda q(2\varphi_1+\lambda\varphi_2)]-2\lambda p\beta_1\delta\gamma^*(\lambda)}{2[\delta q\gamma^*(\lambda)+(\alpha-\delta)q(1+\lambda\varphi_1)]} + \frac{2\{1-\gamma^*(\lambda)\}[\alpha(\lambda\beta_1+p)+\bar{\alpha}(1+\lambda\varphi_1)] +\lambda^2(\alpha\beta_2+\bar{\alpha}\varphi_2)+2\lambda(p\alpha\beta_1+\bar{\alpha}\varphi_1)}{2[\gamma^*(\lambda)-\alpha(\lambda\beta_1+p)-\bar{\alpha}(1+\lambda\varphi_1)]} - \frac{\alpha q\psi_1W(0,z)[(\alpha-\delta)\{2(1+\lambda\varphi_1)[q(1-\gamma^*(\lambda))-\lambda p\beta_1] +\lambda q(2\varphi_1+\lambda\varphi_2)\}-2\lambda p\beta_1\delta\gamma^*(\lambda)]}{2\{\gamma^*(\lambda)-\alpha(\lambda\beta_1+p)-\bar{\alpha}(1+\lambda\varphi_1)\}\{\delta q\gamma^*(\lambda)+(\alpha-\delta)q(1+\lambda\varphi_1)\}} + \frac{\lambda\psi_2q\alpha W(0,z)+2\alpha\psi_1W(o,z)[q(1-\gamma^*(\lambda))-\lambda p\beta_1]}{2\{\gamma^*(\lambda)-\alpha(\lambda\beta_1+p)-\bar{\alpha}(1+\lambda\varphi_1)\}}$$

Let W_S be the average time a customer spends in the system under steady state. Due to little's formula, we have

$$W_S = \frac{L_S}{\lambda}$$

4. Stochastic Decomposition

Stochastic decomposition has been widely observed among M/G/1 type queues with server vacations (see, for example, [10–13]). A key result, in these analyses is that the number of customers in the system in steady state at a random point in time is distributed as the sum of two independent random variables, one of which is the number of customers in the corresponding standard M/G/1 queuing system (in steady state) at a random point in time. The other random variable may have different probabilistic interpretations in specific cases depending on how the vacations are scheduled. Stochastic decomposition has also been observed to hold for some M/G/1 retrial queues [14,15].

Let $\Pi(z)$ be the probability generating function of the number of customers in the M/G/1 queuing system with Bernoulli feedback (see [16]) in steady state at a random point in time, $\chi(z)$ be the probability generating function of the number of customers in the vacation system at a random point in time given that the server is idle and $K(z)$ be the probability generating function of the ran-

dom variable being decomposed. Then, the mathematical version of the stochastic decomposition law is

$$K(z) = \Pi(z) \cdot \chi(z). \tag{4.1}$$

We now verify that the decomposition law applied to the retrial model analyzed in this paper. For the M/G/1 queue with Bernoulli feedback queuing system [16], we have

$$\Pi(z) = \frac{(q-\lambda\beta_1)(1-z)\beta^*(\lambda(1-z))}{(pz+q)\beta^*(\lambda(1-z))-z} \tag{4.2}$$

To obtain an expression for $\chi(z)$ we first define idle in our context. We say that the server is idle if the server is either under repair or free or on vacation. (Note that in retrial queues, there may be customers in the system even when the server is idle). Under this definition, we have

$$\chi(z) = \frac{P_0 + P(z) + R(z) + W(z)}{P_0 + P(1) + R(1) + W(1)}$$

Using the results (3.52), (3.54), (3.55) and (3.57) of theorem 3.2, we obtain

$$\begin{aligned}\chi(z) = {} & \frac{P_0\{[z+(1-z)\gamma^*(\lambda)][z\varphi^*(\lambda(1-z))(\alpha-\delta)-\alpha]}{+\delta z\}\{(pz+q)\beta^*(\lambda(1-z))-z\}} \\ & \frac{}{(1-\frac{\lambda\beta_1}{q})(1-z)\{z-[(pz+q)\alpha\beta^*(\lambda(1-z))} \\ & +z\bar{\alpha}\varphi^*(\lambda(1-z))][z+(1-z)\gamma^*(\lambda)]\} \\ & \alpha[1-\psi^*(\lambda(1-z))][z+(1-z)\gamma^*(\lambda)] \\ & -\frac{[(pz+q)\beta^*(\lambda(1-z))-z]W(0,z)}{\lambda(1-\frac{\lambda\beta_1}{q})(1-z)\{z-[(pz+q)\alpha\beta^*(\lambda(1-z))} \\ & +z\bar{\alpha}\varphi^*(\lambda(1-z))][z+(1-z)\gamma^*(\lambda)]\}\end{aligned} \tag{4.3}$$

From (2.56) we observe that $K(z) = \Pi(z) \cdot \chi(z)$ which confirms that the decomposition law of [6] is also valid for this special vacation system.

References

[1] Choi, B. D. and Kulkarni, V. G., "Feedback Retrial Queuing Systems," *Stochastic Model Relat. Field,* pp. 93–105 (1992).

[2] Kulkarni, V. G. and Choi, B. D., "Retrial Queue with Server Subject to Breakdown and Repairs," *Queuing Syst.,* Vol. 7, pp. 191–208 (1990).

[3] Choi, B. D., Shin, Y. W. and Ahn, W. C., "Retrial Queues with Collision Arising from Unslotted CSMA/CD Protocol," *Queuing syst.,* Vol. 11, pp. 335–356 (1992).

[4] Yang, T. and Li, H., "The M/G/1 Retrial Queue with the Server Subject to Starting Failures," *Queuing Syst.,* Vol. 16, pp. 83–96 (1994).

[5] Fayolle, G., "A Simple Telephone Exchange with Delayed Feedbacks, in: O. J. Boxma, J. W. Cohen, H. C. Tijms (Eds.)," *Traffic Analysis and Computer Performance Evaluation, Elsevier, Amsterdam,* pp. 245–253 (1986).

[6] Gomez-Corral, A., "Stochastic Analysis of a Single Server Retrial Queue with General Retrial Times," *Nav. Res. Logis.,* Vol. 46, pp. 561–581 (1999).

[7] Krishna Kumar, B., Pavai Madheswari, S. and Vijaya-Kumar, A., "The M/G/1 Retrial Queue with Feedback and Starting Failures," *Applied Mathematical Modelling,* Vol. 26, pp. 1057–1075 (2002).

[8] Lee, H. W., Lee, S. S., Chae, K. C. and Nadarajan, R., "On a Batch Service Queue with Single Vacation," *Appl. Math Modeling*, Vol. 16, pp. 36–42 (1992).

[9] Choi, B. D., Rhee, K. H. and Park, K. K., "The M/G/1 Retrial Queue with Retrial Rate Control Policy," *Prob. Eng. Info. Sci.,* Vol. 7, pp. 26–46 (1993).

[10] Cooper, R. B., "Queues Served in Cyclic Order: Waiting Times," *Bell Syst. Tech.,* Vol. 49, pp. 339–413 (1970).

[11] Doshi, B. T., "A Note on Stochastic Decomposition in a GI/G/1 Queue with Vacation or Setup Times," *J. Appl. Prob.,* Vol. 22, pp. 419–428 (1985).

[12] Fuhrmann, S. W. and Cooper, R. B., "Stochastic Decomposition in the M/G/1 Queue with Generalized Vacations," *Opns. Res.,* Vol. 33, pp. 1117–1129 (1985).

[13] Levy, Y. and Yechiali, U., "Utilization of Idle Time in an M/G/1 Queuing System," *Manage. Sci.,* Vol. 22, pp. 202–211 (1975).

[14] Artalejo, J. R. and Gomez-Corral, A., "Steady State

Solution of a Single Server Queue with Linear Request Repeated," *J. Appl. Prob.,* Vol. 34, pp. 223–233 (1997).

[15] Yang, T. and Templeton, J. G. C., "A Survey on Retrial Queue," *Queuing Syst.,* Vol. 2, pp. 201–233 (1987).

[16] Takacs, L., "A Single Server Queue with Feedback," *Bell Syst. Tech. J.,* Vol. 42, pp. 505–519 (1963).

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